

On the dualization of scalars into $(d - 2)$ -forms in supergravity

Momentum maps, R-symmetry and gauged supergravity

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Abstract

We review and investigate different aspects of scalar fields in supergravity theories both when they parametrize symmetric spaces and when they parametrize spaces of special holonomy which are not necessarily symmetric (Kähler and Quaternionic-Kähler spaces): their rôle in the definition of derivatives of the fermions covariant under the R-symmetry group and (in gauged supergravities) under some gauge group, their dualization into $(d - 2)$ -forms, their role in the supersymmetry transformation rules (via fermion shifts, for instance) etc. We find a general definition of momentum map that applies to any manifold admitting a Killing vector and coincides with those of the holomorphic and tri-holomorphic momentum maps in Kähler and quaternionic-Kähler spaces and with an independent definition that can be given in symmetric spaces. We show how the momentum map occurs ubiquitously: in gauge-covariant derivatives of fermions, in fermion shifts, in the supersymmetry transformation rules of $(d - 2)$ -forms etc. We also give the general structure of the Noether-Gaillard-Zumino conserved currents in theories with fields of different ranks in any dimension.

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Introduction

One of the main features of supergravity theories is the presence of scalar fields. In many cases this presence can be traced to a compactification of some higher-dimensional supergravity theory and, then, the scalars encode a great deal of information about the moduli space of the compactification. In gauged supergravity theories, the scalar potential gives rise to symmetry and supersymmetry breaking and identify possible vacua, and can be used to construct inflationary models. Furthermore, the gravitating solutions of supergravity theories can (or must, depending on the case) have active scalars. For instance, the supergravity generalizations of the Reissner-Nordström black hole have them, giving rise to very interesting phenomena such as the attractor mechanism [1, 2, 3, 4]. Actually, the fact that their values at infinity (which can be interpreted as their vacuum expectation values) do not occur in the charged black-hole entropy formula [5] is, certainly, a major indication of the existence of a microscopic interpretation for the black-hole entropy. On top of this, the relation of the scalars with compactification moduli plays a fundamental rôle in the microscopic interpretation of the black-hole entropy in the context of string theory [6]. But there are solutions much more directly related to scalar fields: these are the domain walls and the $(d - 3)$ -brane solutions. We will discuss the latter later.

Scalar fields are, therefore, not just a nuisance one has to live with in supergravity, but a blessing, a fantastic tool whose use one has to master, in spite of the fact that, so far, we have only found one scalar field in Nature.

Scalar fields can be coupled non-linearly among themselves (non-linear σ -models and scalar potentials) or to other fields (scalar-dependent kinetic matrices) in very simple ways, without having to include terms of higher-orders in derivatives, because their transformations do not contain spacetime derivatives (even if they couple to a gauge field). Using this property one can rewrite theories of higher-order in derivatives of other fields (for instance, theories of gravity with corrections of higher-order in curvature) as standard quadratic theories with couplings to scalar fields. A well-known example is the equivalence between $f(R)$ theories of gravity and Jordan-Brans-Dicke scalar-tensor theories of gravity.

This versatility of scalar fields comes at a price, though, and the non-linearities create their own problems. In this paper we want to address specially one of them: that of the dualization of scalars into $(d - 2)$ -form potentials.

As it is well known, supergravity theories, as the low-energy, effective field-theory limits of superstring theories, contain a great deal of information about the p -dimensional extended objects (*branes*) that occur in the latter. This information is encoded in the $(p + 1)$ -form potentials they electrically couple to. For $p \leq d/2 - 2$, these fields appear in the supergravity action as fundamental fields. For higher values, though, one has to consider their electric-magnetic duals.

In most cases, the supergravity theory cannot be completely reformulated in terms of the dual supergravity fields: even their field strengths can only be defined using the fundamental ones. We have learned to deal with all of them at the same time using the

so-called “democratic formulations” [7] or PST-type duality-symmetric actions [8, 9] constructed with the use of Pasti-Sorokin-Tonin approach [11, 12].¹

The technical reason is that, typically, the supergravity action contains potentials without derivatives and the standard procedure for dualization requires, as a first step, the replacement of the potentials by their field strengths as independent variables in the action. This problem is more acute for scalar fields, because they generically appear without derivatives in the σ -model metric and in the kinetic matrices.

When the σ -model metric admits an isometry, it is possible to make it independent of the associated scalar coordinate by a change of variables. If the isometry is a global symmetry of the theory, the kinetic matrices may also be independent of that scalar too and, then, one could dualize it into a $(d - 2)$ -form potential. One could repeat the procedure for additional commuting isometries but most σ -models do not have as many commuting isometries as scalar fields, even if they have more isometries than scalar fields, as it happens in $\mathcal{N} > 2, d = 4$ supergravities, whose scalars parametrize symmetric Riemannian spaces. How should one proceed in that case?

An additional problem is that we expect the dualization procedure to preserve all the duality symmetries of the theory, *i.e.* all the symmetries of the equations of motion, including those that do not leave the action invariant. This implies that it is not enough to dualize the scalars (even if possible), since they do not transform in linear representations of the duality group and the dual $(d - 2)$ -form potentials can only transform linearly.

The basic idea to solve this problem was proposed in Ref. [13] for the case of the $SL(2, \mathbb{R})/SO(2)$ σ -model that occurs in $\mathcal{N} = 2B, d = 10$ supergravity (as well as in many other theories): the objects to be dualized are not the scalars but the Noether 1-forms $j_A = j_{A\mu} dx^\mu$ associated to the symmetry². In a background metric $g_{\mu\nu}$, these are related to the Noether current densities j_A^μ satisfying on-shell the continuity equation

$$\partial_\mu j_A^\mu = 0, \quad (0.1)$$

by

$$j_{A\mu} \equiv \frac{j_A^\nu g_{\nu\mu}}{\sqrt{|g|}}. \quad (0.2)$$

In terms of the Noether 1-forms, the continuity equations take the form

$$d \star j_A = 0, \quad (0.3)$$

and can be locally solved by introducing $(d - 2)$ -form potentials B_A so that

¹Observe that the democratic formulation of Ref. [7] does not have manifest $SL(2, \mathbb{R})$ invariance in the IIB sector because only the RR 6- and 8-forms are considered and they are part of a doublet and a triplet. In this sense it is incomplete. The more complicated PST-type type IIB supergravity action of [9] contains the complete set of higher forms.

²Here and in what follows the indices A, B, C, \dots run over the adjoint representation of the whole global symmetry group.

$$\star j_A = dB_A. \quad (0.4)$$

The $(d - 2)$ -form fields B_A are the duals of the scalars.³ In the kind of theories we are interested in, the numbers of scalars and $(d - 2)$ -forms do not coincide in general because there are more global symmetries than scalars. However, there are constraints to be taken into account that reduce the number of independent dynamical degrees of freedom associated to the latter, as we will see.

When the scalars couple to other fields, these must transform under the global symmetries of the σ -model as well. Some of the transformations may be electric-magnetic dualities and only the equations of motion will be left invariant by them. Accordingly, there are no Noether currents for those transformations. As shown in Ref. [14], in the 4-dimensional case it is always possible to use the Noether-Gaillard-Zumino (NGZ) 1-forms [16] which are conserved on-shell, to define the 2-forms B_A . We will study the higher-dimensional, higher-rank analog of the NGZ 1-forms in Section 4.

In order to describe systematically the procedure, it is convenient to start by reviewing the construction of the metrics, Killing vectors, Vielbeins and connection 1-forms etc. in symmetric spaces, since this is the kind of target spaces that occurs in most extended supergravities. We will do this in Section 1. In the process we will (re-)discover structures which appear in the gauging of the theories (specially in the supersymmetric case) (covariant derivatives, fermion shifts etc.) In particular, we are going to see that in all symmetric spaces there exists a generalization of the holomorphic and triholomorphic momentum maps associated to the Kähler-Hodge and quaternionic-Kähler manifolds of $\mathcal{N} = 1, 2$ supergravities in $d = 4$ dimensions (reviewed in the Appendices) which play exactly the same rôle in the construction of the gauge-covariant derivatives of fermions, in the fermion shifts of the supersymmetry transformation rules of gauged supergravities and also in the supersymmetry transformation rules of the $(d - 2)$ -forms dual to the NGZ currents. These generalizations share the same properties and deserve to be called momentum maps as well.

Furthermore, we are also going to give an even more general definition of momentum map (Section 1.1.2), valid for any manifold admitting one isometry, showing that in symmetric spaces, Kähler-Hodge or quaternionic-Kähler spaces our general definition is equivalent to the standard one. This is one of the main results of this paper.

To end Section 1 we will review some well-known examples which will be useful in what follows.

In Section 2 we address the dualization of the scalars of a symmetric σ -model into $(d - 2)$ -form potentials along the lines explained before. Then, in Section 3 we will consider the case in which the scalars of the symmetric σ -model are coupled to the vector fields of a generic 4-dimensional field theory of the kind considered by Gaillard

³Actually, using the embedding-tensor formalism, it can be argued that the $(d - 2)$ -form potentials of any field theory transform in the adjoint representation of the global symmetry group [14, 15]. Some of the symmetries may not act on the scalars at all but, to simplify matters, we focus here on the symmetries of the scalar σ -model.

and Zumino in Ref. [16], introducing the NGZ current 1-form and studying its dualization into 2-forms. We will also consider there (Section 3.1.1) the general form of the supersymmetry transformations of the 2-forms and the rôle played in them by the momentum map. It is because of this rôle that we expect the tensions of the strings that couple to the 2-forms to be determined by the momentum map (Section 3.1.2). We will also show how the momentum map occurs in the fermion shifts of the supersymmetry transformation rules of the fermions of 4-dimensional extended supergravities (Section 3.1.3).

The higher-dimensional case in which the scalars are also coupled to potentials of different and higher ranks will be considered in Section 4 and we will show through examples that the equation of conservation of the generalized NGZ current 1-form has a universal form.

Our conclusions are contained in Section 5.

1 Review of symmetric σ -models

Let us consider⁴ a homogeneous space M on which the Lie group G acts transitively, and where $H \subset G$, topologically closed, is the isotropy subgroup. Then M is homeomorphic to the coset space G/H of equivalence classes under right multiplication by elements of H $\{gH\}$ (G acts from the left on these equivalence classes) and can be given the structure of a manifold of dimension $\dim G - \dim H$. Furthermore, G can be seen as a principal bundle with base space $M = G/H$, structure group H , and projection $G \rightarrow G/H$.

In any homogeneous space G/H , the Lie algebra of G , as a vector space, can be decomposed as the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, where \mathfrak{h} is the Lie subalgebra of H and \mathfrak{k} is its orthogonal complement. By definition of subalgebra

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \quad (1.1)$$

G/H is said to be a reductive homogenous space if

$$[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{k}, \quad (1.2)$$

which means that \mathfrak{k} is a representation space of H . Finally, G/H is said to be symmetric and $(\mathfrak{k}, \mathfrak{h})$ is called a symmetric pair if it is reductive and

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \quad (1.3)$$

The two components of a symmetric pair are mutually orthogonal with respect to the Killing metric which is block-diagonal.

⁴In this review we follow Refs. [17, 18] although a big part of material can be also found in the classical papers Refs. [19, 20, 21, 22].

Now, if G/H is a symmetric space (G connected, and H compact) and there is a G -invariant metric defined on it, then it is Riemannian symmetric space.

The metrics of the scalar σ -models that appear in all supergravities in $d \geq 4$ dimensions with more than 8 supercharges are the metrics of some Riemannian symmetric space. The metrics and the kinetic terms can be constructed using a G/H coset representative or by using a generic element of G and gauging an H subgroup. Let us start by reviewing the first method.

1.1 Coset representative

Let us introduce some notation: we denote by $\{T_A\}$, $\{M_i\}$ and $\{P_a\}$ (where $A, B, \dots = 1, \dots, \dim G$, $i, j, \dots = 1, \dots, \dim H$ and $a, b, \dots = 1, \dots, d \equiv \dim G - \dim H$), three bases of, respectively, \mathfrak{g} , \mathfrak{h} and \mathfrak{k} with $\{T_A\} = \{M_i\} \cup \{P_a\}$ (see Ref. [23, 24] and also Ref. [25]). The structure constants are defined by

$$[T_A, T_B] = f_{AB}{}^C T_C, \quad (1.4)$$

and, by definition of symmetric space, the only non-vanishing components are

$$[M_i, M_j] = f_{ij}{}^k M_k, \quad [P_a, M_i] = f_{ai}{}^b P_b, \quad [P_a, P_b] = f_{ab}{}^i M_i. \quad (1.5)$$

The adjoint representation of \mathfrak{g} is defined by the matrices

$$\Gamma_{\text{Adj}}(T_A)^B{}_C \equiv f_{AC}{}^B, \quad (1.6)$$

and, obviously, their restriction to the indices i, j is the adjoint representation of \mathfrak{h} . Furthermore, the matrices

$$\Gamma(M_i)^a{}_b = f_{ib}{}^a, \quad (1.7)$$

provide another representation of \mathfrak{h} with representation space \mathfrak{k} .

Only the diagonal blocks of the the Killing metric

$$K_{AB} \equiv \text{Tr} [\Gamma_{\text{Adj}}(T_A) \Gamma_{\text{Adj}}(T_A)] = f_{AC}{}^D f_{BD}{}^C, \quad (1.8)$$

K_{ab} and K_{ij} are non-vanishing ($K_{ai} = 0$). K_{ij} is the restriction of the Killing metric⁵ of G to H and, for the kind of groups we are considering, it is proportional to the Killing metric of H .⁶ Under the adjoint action of G , defined by

$$g^{-1} T_A g \equiv T_B \Gamma_{\text{Adj}}(g^{-1})^B{}_A, \quad (1.9)$$

The Killing metric metric is invariant due to the cyclic property of the trace

⁵That is: $K_{ij} = f_{iC}{}^D f_{jD}{}^C = f_{ik}{}^l f_{jl}{}^k + f_{ia}{}^b f_{jb}{}^a$.

⁶Because $f_{ia}{}^b f_{jb}{}^a = \text{Tr}[\Gamma(M_i) \Gamma(M_j)] \propto f_{ik}{}^l f_{jl}{}^k$.

$$K_{AB} = \text{Tr} \left[\Gamma_{\text{Adj}}(g^{-1} T_A g g^{-1} T_B g) \right] = K_{CD} \Gamma_{\text{Adj}}(g^{-1})^C{}_A \Gamma_{\text{Adj}}(g^{-1})^D{}_B. \quad (1.10)$$

Then, since the Killing metric is invertible (K^{AB}) for the kind of groups we are considering, we find that

$$K_{BC} \Gamma_{\text{Adj}}(g)^C{}_D K^{DA} = \Gamma_{\text{Adj}}(g^{-1})^A{}_B. \quad (1.11)$$

Let us denote by $u(\phi) = u(\phi^1, \dots, \phi^d)$ a coset representative of G/H in some local coordinate patch. In practice it will be a matrix transforming in some representation r of G . The scalar fields of the σ -model will be mappings from spacetime to G/H expressed in these coordinates as the functions $\phi^m(x)$. Under a left transformation $g \in G$, $u(\phi)$ transforms into another element of G , which becomes a coset representative $u(\phi')$ only after a right transformation with the inverse of $h \in H$, that is (see Refs. [19, 20, 21, 22])

$$gu(\phi) = u(\phi')h. \quad (1.12)$$

For a given choice of coset representative u , h will depend on g and ϕ , but we will not indicate explicitly that dependence.

The left-invariant Maurer–Cartan 1-form $V \in \mathfrak{g}$ and can be expanded as follows:⁷

$$V \equiv -u^{-1}du = e^a P_a + \vartheta^i M_i. \quad (1.13)$$

The e^a components can be used as Vielbeins in G/H and the ϑ^i components play the role of connection⁸. The Maurer–Cartan equations satisfied by V ($dV - V \wedge V = 0$) take the following form in terms of the above 1-form components:

$$de^a - \vartheta^i \wedge e^b f_{ib}{}^a = 0, \quad (1.14)$$

$$d\vartheta^i - \frac{1}{2} \vartheta^j \wedge \vartheta^k f_{jk}{}^i - \frac{1}{2} e^b \wedge e^c f_{bc}{}^i = 0. \quad (1.15)$$

Comparing the first of these equations with Cartan’s structure equation with vanishing torsion $\mathcal{D}e^a = de^a + \omega_b{}^a \wedge e^b = 0$ we find the connection 1-form

$$\omega_b{}^a = -\vartheta^i f_{ib}{}^a, \quad (1.16)$$

which also justifies the identification of ϑ^i with a connection. The curvature 2-form of this connection is, from the definition

⁷The elements of the basis of \mathfrak{k} and \mathfrak{h} will be in the representation r in which u transforms. However, for the sake of simplicity, we will write P_a instead of $\Gamma_r(P_a)$ etc. whenever this does not lead to confusion.

⁸See the transformation rules Eqs. (1.21).

$$R_b{}^a(\omega) = -R(\vartheta)^i f_{ib}{}^a, \quad \text{where} \quad R(\vartheta)^i \equiv d\vartheta^i - \frac{1}{2}\vartheta^j \wedge \vartheta^k f_{jk}{}^i. \quad (1.17)$$

Then, Eqs. (1.15) tell us that

$$R(\vartheta)^i = \frac{1}{2}e^b \wedge e^c f_{bc}{}^i, \quad R_b{}^a(\omega) = -\frac{1}{2}e^d \wedge e^e f_{de}{}^i f_{ib}{}^a. \quad (1.18)$$

We have defined a Vielbein basis and an affine connection on G/H , but we have not defined a Riemannian metric yet. We can do so if we are provided with a metric g_{ab} on \mathfrak{k} :

$$ds^2 = g_{ab}e^a \otimes e^b = g_{ab}e^a{}_m e^b{}_n d\phi^m d\phi^n \equiv \mathcal{G}_{mn}(\phi) d\phi^m d\phi^n. \quad (1.19)$$

Its pullback over spacetime, conveniently normalized, can be used in the action for the σ -model:

$$S = \frac{1}{2} \int \mathcal{G}_{mn}(\phi) d\phi^m \wedge \star d\phi^n. \quad (1.20)$$

In order to construct a Riemannian symmetric σ -model the metric $\mathcal{G}_{mn}(\phi)$ must be invariant under the left action of G . Under the left multiplication by $g \in G$, $u(\phi') = gu(\phi)h^{-1}$, and the components of the left-invariant Maurer–Cartan 1-form transform in the adjoint representation of H (the ϑ^i as a connection of H):

$$\begin{cases} e^a(\phi') &= (he(\phi)h^{-1})^a = \Gamma_{\text{Adj}}(h)^a{}_b e^b(\phi), \\ \vartheta^i(\phi') &= (h\vartheta(\phi)h^{-1})^i + (dh h^{-1})^i, \end{cases} \quad (1.21)$$

where $e(\phi) = e^a(\phi)P_a$ and $\vartheta(\phi) = \vartheta^i(\phi)M_i$. Infinitesimally,

$$h \sim 1 + \sigma^i(\phi)M_i, \quad \Rightarrow \quad e^a(\phi') \sim e^a(\phi) + \sigma^i(\phi)f_{ib}{}^a e^b(\phi), \quad (1.22)$$

and the Riemannian metric $\mathcal{G}_{mn}(\phi)$ will be invariant under the left action of G if the metric g_{ab} on \mathfrak{k} is H -invariant:

$$f_{i(a}{}^c g_{b)c} = 0. \quad (1.23)$$

In all the relevant cases we can set $g_{ab} \propto K_{ab}$, the projection on \mathfrak{k} of the Killing metric and we will do so from now on. More precisely, we will use this normalization:⁹

$$g_{ab} = \text{Tr}[P_a P_b]. \quad (1.24)$$

Observe that the H -invariance of g_{ab} Eq. (1.23) automatically guarantees that the torsionless connection $\omega_b{}^a$ in Eq. (1.16) is metric-compatible and, therefore, it is the Levi–Civita connection of the above metric.

⁹We will sometimes use $g_{AB} = \text{Tr}[T_A T_B]$.

In addition to the isometries corresponding to the left action of G , with Killing vectors k_A , the resulting Riemannian metric is also invariant under the right action of $N(H)/H$, $N(H)$ being the normalizer of H in G . The Killing vectors associated to the latter are just the vectors e_a dual to the horizontal Maurer–Cartan 1-forms in the directions of $N(H)/H$ [17].

Our next task is to find the general expression of the Killing vectors k_A which defines the transformation rule of the Goldstone fields. From the infinitesimal version of $gu(\phi) = u(\phi')h$ with

$$\begin{aligned} g &= 1 + \sigma^A T_A, \\ h &= 1 - \sigma^A W_A^i M_i, \\ \phi^{m'} &= \phi^m + \sigma^A k_A^m, \end{aligned} \tag{1.25}$$

where W_A^i is known as the *H-compensator*, we get after some straightforward manipulations

$$k_A^a = -\Gamma_{\text{Adj}}(u^{-1}(\phi))^a{}_A, \tag{1.26}$$

$$W_A^i = -k_A^m \vartheta_m^i - P_A^i, \tag{1.27}$$

where we have defined the *momentum map* P_A^i

$$P_A^i \equiv \Gamma_{\text{Adj}}(u^{-1}(\phi))^i{}_A. \tag{1.28}$$

It transforms as

$$P_A'^i = \Gamma_{\text{Adj}}(h(\phi))^i{}_j P_B^j \Gamma_{\text{Adj}}(g^{-1})^B{}_A, \tag{1.29}$$

and it plays a crucial role in the gauging of the global symmetry group G , as we are going to see below.

1.1.1 H-covariant derivatives and the momentum map

With the objects that we have found we can construct *H-covariant derivatives* and *H-covariant Lie derivatives*, which transform covariantly under the compensating H transformations associated to global G transformations. Let us start by discussing the former.

In supergravity theories, H coincides with the R -symmetry group, under which all the fermions transform, and, therefore, all the derivatives of fermions must be H -covariant derivatives. Under a global G transformation of the scalars, these spacetime fields undergo an H scalar-dependent, compensating transformation that can be contravariant $\tilde{\zeta}' = \Gamma_s(h)\tilde{\zeta}$, or covariant, $\psi' = \psi\Gamma_s(h^{-1})$, in some representation s of H .

For those fields, with the help of the pullback over the spacetime of the H connection $\vartheta_m^i d\phi^m$,¹⁰ (see the second of Eqs. (1.21)), we define the H-covariant¹¹ derivative by

$$\mathcal{D}\xi \equiv d\xi - \vartheta^i \Gamma_s(M_i)\xi, \quad \mathcal{D}\psi \equiv d\psi + \psi \vartheta^i \Gamma_s(M_i). \quad (1.30)$$

The H-covariant derivative satisfies the Ricci identities

$$\mathcal{D}^2 \xi = -R(\vartheta)^i \Gamma_s(M_i)\xi, \quad \mathcal{D}^2 \psi = \psi R(\vartheta)^i \Gamma_s(M_i), \quad (1.31)$$

where $R(\vartheta)^i$ stands here for the curvature 2-form defined in Eq. (1.18).

Using

$$k_A^a k_B^b f_{ab}^i + P_A^j P_B^k f_{jk}^i = \Gamma_{\text{Adj}}(u^{-1})^{A'}{}_A \Gamma_{\text{Adj}}(u^{-1})^{B'}{}_B f_{A'B'}^i = f_{AB}{}^C P_C^i, \quad (1.32)$$

we find that the momentum map that we have defined above satisfies the equivariance condition

$$\mathcal{D}_A P_B^i - \mathcal{D}_B P_A^i - k_A^a k_B^b f_{ab}^i + P_A^j P_B^k f_{jk}^i = f_{AB}{}^C P_C^i, \quad (1.33)$$

where $\mathcal{D}_A = k_A^m \mathcal{D}_m$. Using the explicit form of the curvature Eq. (1.18) it is easy to derive the following equation, which is sometimes used as definition of the momentum map

$$\mathcal{D}_m P_A^i = -R_{mn}{}^i(\vartheta) k_A^n. \quad (1.34)$$

One is often interested in gauging the global symmetry group G (or a subgroup of G), making the supergravity theory invariant under transformations of the form Eq. (1.25) with parameters σ^A which are promoted to arbitrary spacetime functions $\sigma^A(x)$. Under these transformations, the pullback of the second equation of (1.21) acquires an additional term and the (spacetime pullback of) above H-covariant derivatives do not transform covariantly anymore. As usual, it is necessary to introduce spacetime 1-forms A^A and modify the above covariant derivatives as follows:¹²

$$\mathfrak{D}\xi \equiv d\xi - \left(A^A P_A^i + \vartheta^i \right) \Gamma_s(M_i)\xi, \quad \mathfrak{D}\psi \equiv d\psi + \psi \left(A^A P_A^i + \vartheta^i \right) \Gamma_s(M_i). \quad (1.35)$$

¹⁰We will denote the pullback with the same symbol, ϑ^i when this does not lead to confusion.

¹¹In spite of the name, which is, admittedly, misleading, there is no true gauge symmetry in this construction. The H-transformations are not arbitrary functions of the spacetime coordinates. Neither they are arbitrary functions of the scalar fields (the coordinates on G/H). The only arbitrary parameters in these transformations are the global parameters of the G transformation that needs to be compensated to go back to the coset representative.

¹²Here we will not concern ourselves with the problem of matching the rank of the subgroup of G to be gauged with the number of 1-forms available in the supergravity theory. In general this requires the explicit or implicit introduction of the so-called *embedding tensor*. We will use it in Section 3.1.3.

The structure of these gauge covariant derivatives is identical to the covariant derivatives that occur in gauged $\mathcal{N} = 1, 2$ supergravities¹³, even if the scalar manifolds (Kähler–Hodge and quaternionic–Kähler manifolds) are no coset spaces: the spinors of these theories transform under $U(1)$ Kähler transformations and the Kähler 1-form connection plays the role of ϑ in Eq. (1.30). In $\mathcal{N} = 2$ theories with hypermultiplets, the spinors also transform under $SU(2)$ compensating transformations and the pullback of the $SU(2)$ connection of the quaternionic–Kähler manifold plays the role of ϑ . Associated to these symmetries there are holomorphic and tri-holomorphic momentum maps which play the same role as P_A^i . A more detailed comparison between these structures and the ones that arise in symmetric spaces can be found in the appendices.

Observe that these covariant derivatives cannot be obtained by the often-used (but generally wrong) replacement of the pullback by the “covariant pullback” of the H connection ϑ^i in Eqs. (1.30)

$$\vartheta_m^i d\phi^m \rightarrow \vartheta_m^i \mathfrak{D}\phi^m, \quad (1.36)$$

where

$$\mathfrak{D}\phi^m \equiv d\phi^m - A^A k_A^m, \quad (1.37)$$

is the covariant derivative of the scalars, because, according to Eq. (1.27),

$$\vartheta_m^i \mathfrak{D}\phi^m = \left(A^A P_A^i + \vartheta_m^i d\phi^m \right) + A^A W_A^i, \quad (1.38)$$

and the H-compensator does not vanish in general. Something similar happens in the Kähler–Hodge and quaternionic–Kähler manifolds of $\mathcal{N} = 1, 2$, $d = 4$ supergravities [26, 18].

Using the identity Eq. (1.34) and other results derived in this section one can compute the Ricci identities for the \mathfrak{D} covariant derivative

$$\begin{aligned} \mathfrak{D}^2 \xi &= - \left[F^A P_A^i + R(\vartheta)_{mn}^i \mathfrak{D}\phi^m \mathfrak{D}\phi^n \right] \Gamma_s(M_i) \xi, \\ \mathfrak{D}^2 \psi &= \psi \left[F^A P_A^i + R(\vartheta)_{mn}^i \mathfrak{D}\phi^m \mathfrak{D}\phi^n \right] \Gamma_s(M_i), \end{aligned} \quad (1.39)$$

where

$$F^A = dA^A - \frac{1}{2} f_{BC}^A A^B \wedge A^C. \quad (1.40)$$

1.1.2 A more basic definition of the momentum map

Let us consider a d Riemannian manifold M ,¹⁴ not necessarily symmetric, but admitting a set of Killing vectors k_A^a , that is

¹³See, for instance, Refs. [26, 27, 18].

¹⁴The signature is irrelevant in this discussion, which also applies to pseudo-Riemannian spaces.

$$\nabla_{(a|}k_{A|b)} = 0, \quad (1.41)$$

where ∇_a is the Levi-Civita covariant derivative. To each Killing vector one can associate an infinitesimal rotation in tangent space generated by

$$P_A{}^b{}_a \equiv \nabla_a k_A{}^b. \quad (1.42)$$

The antisymmetry of $P_{Aab} = g_{ac}P_A{}^c{}_a$ follows from the Killing equation. Let $\{M_i\}$ be a basis of the Lie algebra of the holonomy group of M. Generically it will be $\text{SO}(d)$ but for spaces of special holonomy it will be some subgroup $H \subset \text{SO}(d)$. In particular, for Riemannian homogeneous spaces G/H , the holonomy group is precisely H . We can, then, decompose $P_A{}^b{}_a$ in that basis, defining at the same time the coefficients as the components of the momentum map

$$P_A{}^b{}_a \equiv P_A{}^i \Gamma(M_i)^b{}_a. \quad (1.43)$$

It is not hard to show using the explicit expressions for the Killing vectors and connection Eqs. (1.26) and (1.16) that the momentum map we have just defined reduces to the one defined in Eq. (1.28) for symmetric spaces. Furthermore, using the general identity for Killing vectors

$$\nabla_a \nabla_b k_A{}^c = k_A{}^d R_{dab}{}^c, \quad \Rightarrow \quad \nabla_a P_A{}^c{}_b = k_A{}^d R_{dab}{}^c, \quad (1.44)$$

and decomposing both sides of this equation in the basis of the holonomy algebra¹⁵, we get a general version of Eq. (1.34)

$$\nabla_a P_A{}^i = k_A{}^d R_{da}{}^i(\vartheta). \quad (1.46)$$

Finally, we can also show that the momentum map satisfies the equivariance property

$$[P_A, P_B]^a{}_b = f_{AB}{}^C P_C{}^a{}_b - k_A{}^c k_B{}^d R_{cda}{}^b. \quad (1.47)$$

Under an infinitesimal isometry

$$\delta_\alpha x^m = \alpha^A k_A{}^m, \quad (1.48)$$

objects living in tangent space (vectors ψ^a , say¹⁶) transform as

$$\delta_\alpha \psi^a = \alpha^A (P_A{}^a{}_b + k_A{}^m \omega_m{}^a{}_b) \psi^b, \quad (1.49)$$

¹⁵That is

$$R_{adb}{}^c = -R_{ad}{}^i \Gamma(M_i)^c{}_b, \quad (1.45)$$

where the minus sign is chosen to match the sign of the H-curvature $R(\vartheta)$ in symmetric spaces.

¹⁶These variables arise naturally in $\mathcal{N} = 1$ supersymmetric mechanics where one introduces d scalar multiplets $x^m + \theta e^m{}_a \psi^a$. See, e.g. [28, 29] and references therein.

and, when gauging the isometry group, these compensating transformations must be taken into account in the construction of the covariant derivative, which is given by

$$\mathfrak{D}_m \psi^a = \nabla_m \psi^a - A^A{}_m P_A{}^a{}_b \psi^b = \partial_m \psi^a - \left(A^A{}_m P_A{}^a{}_b + \omega_m{}^a{}_b \right) \psi^b. \quad (1.50)$$

This covariant derivative reduces to the H-covariant derivative in Eq. (1.35) for symmetric spaces. The above definition can be generalized to objects transforming in other representations of $\text{SO}(n)$ in the obvious way.

1.1.3 H-covariant Lie derivatives

The H-compensator can be understood as the “local” parameter of the H compensating transformation associated to the infinitesimal G transformation generated by T_A or the Killing vector k_A . To study the behavior under G global transformations of fields transforming under these H compensating transformations (something usually done through the Lie derivative) it is necessary to take into account the latter. This requires an H-covariant generalization of the standard Lie derivative with respect to the Killing vector k_A denoted by \mathbb{L}_{k_A} .¹⁷ The equivariance property of the H-compensator

$$\mathcal{L}_{k_A} W_B{}^i - \mathcal{L}_{k_B} W_A{}^i + W_A{}^j W_B{}^k f_{jk}{}^i = f_{AB}{}^C W_C{}^i, \quad (1.51)$$

plays an essential role.

On fields transforming contravariantly $\xi' = \Gamma_s(h)\xi$, or covariantly $\psi' = \psi\Gamma_s(h^{-1})$, the H-covariant derivative is defined by

$$\mathbb{L}_{k_A} \xi \equiv \mathcal{L}_{k_A} \xi + W_A{}^i \Gamma_s(M_i) \xi, \quad \mathbb{L}_{k_A} \psi \equiv \mathcal{L}_{k_A} \psi - \psi W_A{}^i \Gamma_s(M_i). \quad (1.52)$$

The main properties satisfied by this derivative are

$$[\mathbb{L}_{k_A}, \mathbb{L}_{k_B}] = \mathbb{L}_{[k_A, k_B]}, \quad (1.53)$$

$$\mathbb{L}_{k_A} e^a = 0, \quad (1.54)$$

$$\mathbb{L}_{k_A} u = \mathcal{L}_{k_A} u - u W_A{}^i M_i = T_A u. \quad (1.55)$$

Infinitesimally, the H connection ϑ^i transforms with the H-covariant derivative of the transformation parameters. Thus, an appropriate definition for its H-covariant Lie derivative would be

$$\mathbb{L}_{k_A} \vartheta^i \equiv \mathcal{L}_{k_A} \vartheta^i + \mathcal{D} W_A{}^i. \quad (1.56)$$

¹⁷See Ref. [18], which we follow here, and references therein. One could define the H-covariant Lie derivative with respect to any vector but the crucial Lie-algebra property Eq. (1.53) only holds for Killing vectors.

Using the definitions and Eq. (1.34), one can show that

$$\mathbb{L}_{k_A} \vartheta^i = 0. \quad (1.57)$$

1.1.4 Final remarks

The H-covariant derivative of $u(\phi)$, which transforms covariantly in some representation r , is, according to the definition

$$\mathcal{D}u \equiv du + u\vartheta^i M_i = u \left[u^{-1} du + u\vartheta^i M_i \right] = -ue^a P_a, \quad (1.58)$$

where we have used the expansion of the Maurer–Cartan 1-form Eq. (1.13).

We can use this result to obtain a very convenient expression of the action of the σ -model directly in terms of the coset representative $u(\phi)$, which transforms linearly under G (with the metric defined in Eq. (1.24)):¹⁸

$$S = \frac{1}{2} \int \text{Tr}[u^{-1} \mathcal{D}u \wedge \star u^{-1} \mathcal{D}u]. \quad (1.60)$$

The invariance under the left action of G on the coset representative is manifest in this form. This expression connects this approach with the approach that we are going to review in the next section.

To end this section, when the coset space is of the kind $SL(n)/SO(n)$, there is an alternative but completely equivalent construction which is often used in supergravity¹⁹. One defines the symmetric and H-invariant matrix

$$\mathcal{M} \equiv uu^T, \quad \mathcal{M}' = \mathcal{M}(\phi') = g\mathcal{M}(\phi)g^T, \quad (1.61)$$

and choosing a basis in which the P_a are symmetric matrices and the M_i are antisymmetric, it is not difficult to show that

$$\text{Tr}[\mathcal{M}^{-1} d\mathcal{M} \wedge \star \mathcal{M}^{-1} d\mathcal{M}] = 2\text{Tr}[u^{-1} \mathcal{D}u \wedge \star u^{-1} \mathcal{D}u]. \quad (1.62)$$

The equations of motion from this action are obtained much more easily from the formulation that we are going to discuss in the next section because one does not have to deal with the scalar dependence of the connection ϑ^i .

¹⁸Using the cyclic property of the trace, it can also be written in the form

$$S = \frac{1}{2} \int \text{Tr}[\mathcal{D}uu^{-1} \wedge \star \mathcal{D}uu^{-1}]. \quad (1.59)$$

¹⁹This coset arises naturally in toroidal compactifications.

1.2 Gauging of an H subgroup

σ -models on coset spaces G/H are often constructed by gauging a subgroup H of a σ -model constructed on the group manifold G . The latter has the action

$$S_G[u] = \frac{1}{2} \int \text{Tr}[u^{-1} du \wedge \star u^{-1} du], \quad (1.63)$$

where $u(\varphi) = u(\varphi^1, \dots, \varphi^{\dim G})$ is now a generic element of the group G in some representation r and in some local coordinate patch that contains the identity. This action is invariant under the left and right (global) action of the group G and, therefore, its global symmetry group is $G \times G$.

Now we want to gauge a subgroup H of the right symmetry group, under which $u' = uh^{-1}(x)$. Here $h(x)$ stands for an arbitrary function of the spacetime coordinates that gives an element of H at each point. Such a function can be constructed by exponentiation of a linear combination the generators of \mathfrak{h} with coefficients $\sigma^i(x)$ which are arbitrary functions of the spacetime coordinates. After gauging, the global symmetry group will be broken to $G \times H$.

We introduce an \mathfrak{h} -valued spacetime gauge field $A = A_\mu^i dx^\mu M_i$ transforming exactly as the ϑ^i components of the left-invariant Maurer–Cartan 1-form of G/H Eq. (1.21) and the H -covariant derivative

$$\mathcal{D}u \equiv du + uA, \quad (1.64)$$

and we simply replace the exterior derivative d by \mathcal{D} in the action Eq. (1.63) without adding a kinetic term for the gauge field:

$$S_{\text{Gauged}}[u, A] = \frac{1}{2} \int \text{Tr}[u^{-1} \mathcal{D}u \wedge \star u^{-1} \mathcal{D}u]. \quad (1.65)$$

The gauged action describes only $d = \dim G - \dim H$ degrees of freedom: on general grounds we expect that the gauge symmetry can be used to eliminate $\dim H$ of the scalars φ^x , $x = 1, \dots, \dim G$, from the action, leaving only those that parametrize the coset space G/H , that we have denoted by ϕ^m .²⁰ This is the so-called *unitary gauge*. As we are going to see, only d of the scalar equations of motion are independent, in complete agreement with the general expectation.

Let us first consider the equations of motion of the gauge field. These are purely algebraic:

$$\frac{\delta S_{\text{Gauged}}}{\delta A^i} = \star \text{Tr}[M_i u^{-1} \mathcal{D}u] = 0, \quad (1.66)$$

and the solution is

²⁰The representation of the coset element by a group element defined modulo H -transformations is also characteristic of the harmonic superspace approach [30, 25] as well as of the spinor moving frame formalism [31, 32]. See Ref. [33] for a recent application and more references.

$$A^i = V^i, \quad (1.67)$$

where $V = -u^{-1}du$ is the left-invariant Maurer–Cartan 1-form in G . In the unitary gauge, V depends only on the physical scalars ϕ^m and becomes, automatically, the left-invariant Maurer–Cartan 1-form in G/H , so $V^i = \vartheta^i$.

This solution can be substituted in the above action: this substitution $A^i = V^i(\phi)$ and the derivation of the equations of motion for the scalars from the action are two operations that commute and the final result is the same. As a matter of fact, after the substitution, the scalar equations of motion are

$$\frac{\delta S_{\text{Gauged}}[\phi, V(\phi)]}{\delta \phi^x} = \frac{\delta S_{\text{Gauged}}[\phi, A]}{\delta \phi^x} \Big|_{A^i=V^i} + \frac{\delta S_{\text{Gauged}}[\phi, A]}{\delta A^i} \Big|_{A^i=V^i} \frac{\delta V^i}{\delta \phi^x}, \quad (1.68)$$

but the second term vanishes identically.

In the unitary gauge, after the substitution $A^i = V^i = \vartheta^i(\phi)$ we recover the action Eq. (1.60). Classically, these two formulations are completely equivalent. However, in this formulation the scalar equations of motion are easier to derive because we have just shown that we can ignore the variations of the connection with respect to the scalars.

The equations of motion of the scalars ϕ^x , $x = 1, \dots, \dim G$ are

$$\frac{\delta S_{\text{Gauged}}}{\delta \phi^x} = 2V_x^A \text{Tr}[T_A \mathcal{D} \star (u^{-1} \mathcal{D} u)] = 0. \quad (1.69)$$

The invariance under local, right, H -transformations implies, according to Noether's second theorem, the following $\dim H$ Noether identities relating the scalar equations of motion:

$$\text{Tr}[M_i \mathcal{D} \star (u^{-1} \mathcal{D} u)] = 0. \quad (1.70)$$

These are off-shell identities and are also valid in the unitary gauge after the substitution $A^i = \vartheta^i$. Therefore, they are valid in the case discussed in the previous section.

Taking into account the gauge identities the only non-trivial equations of motion are those of the physical scalars ϕ^m , which take the form

$$\frac{\delta S_{\text{Gauged}}}{\delta \phi^m} = e^a{}_m \text{Tr}[P_a \mathcal{D} \star (u^{-1} \mathcal{D} u)] = e^a{}_m g_{ab} \mathcal{D} \star e^b = 0. \quad (1.71)$$

1.3 Examples

In this subsection we are going to review a few examples which we will use repeatedly in what follows:

1. The $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ coset space, which occurs in many supergravities: $\mathcal{N} = 2B, d = 10$ supergravity (the effective field theory of the type IIB superstring), $\mathcal{N} = 2, d = 9, 8$ supergravity (obtained from the former by toroidal dimensional reduction), $\mathcal{N} = 4, d = 4$ supergravity (the effective field theory of the heterotic string compactified on a six torus), and in many truncations of the maximal supergravities in diverse dimensions. In $\mathcal{N} = 2B, d = 10$ supergravity the scalar fields that parametrize this coset are the dilaton φ and RR 0-form χ , combined into the axidilaton field $\tau = \chi + ie^{-\varphi}$ (see Eq. (1.76) below). In $\mathcal{N} = 4, d = 4$ supergravity the fields are the 4-dimensional dilaton ϕ and the dual of the 4-dimensional Kalb-Ramond 2-form a and $\tau = a + ie^{-2\phi}$.
2. The $\text{SU}(1, 1)/\text{U}(1)$ coset, which is an often used completely equivalent alternative form of the $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ coset space: constructed by Schwartz in the $\text{SU}(1, 1)$ formulation [34, 35, 36] and rewritten in the $\text{SL}(2, \mathbb{R})$ formulation in which the dilaton and RR 0-form appear more naturally in Ref. [37]. The supersymmetry transformations of all the fields of $\mathcal{N} = 2B, d = 10$ supergravity, including the 8-forms dual to the dilaton and RR 0-form, were given in the $\text{SU}(1, 1)$ formulation in Ref. [38, 39] and [9]. In Section 4.1.3 we are going to study the dualization of the scalars and the supersymmetry transformations of the dual 8-form fields proposed in that reference from the geometrical point of view taken here.
3. The $\text{E}_{7(+7)}/\text{SU}(8)$ coset space of $\mathcal{N} = 8, d = 4$ supergravity.

1.3.1 $\text{SL}(2, \mathbb{R})/\text{SO}(2)$

The group $\text{SL}(2, \mathbb{R})$ is isomorphic to $\text{SO}(2, 1)$. The Lie brackets of the three generators $\{T_A\}$ can be conveniently written in the form

$$[T_A, T_B] = -\varepsilon_{ABD} Q^{DC} T_C, \quad \Rightarrow f_{AB}{}^C = -\varepsilon_{ABD} Q^{DC}, \quad A, B, \dots = 1, 2, 3, \quad (1.72)$$

where $Q = \text{diag}(+ + -)$. A 2-dimensional representation (the one we are going to work with) is provided by

$$T_1 = \frac{1}{2}\sigma^3, \quad T_2 = \frac{1}{2}\sigma^1, \quad T_3 = \frac{i}{2}\sigma^2, \quad (1.73)$$

where the σ^i s are the standard Hermitian, traceless, Pauli matrices satisfying $\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk} \sigma^k$. The Killing metric is

$$K_{AB} = -2Q_{AB}, \quad \text{and} \quad g_{AB} = \text{Tr}(T_A T_B) = \frac{1}{2}Q_{AB}. \quad (1.74)$$

A convenient $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ coset representative is provided by

$$u = e^{-\varphi T_1} e^{e^{\varphi/2} \chi (T_2 + T_3)} = \begin{pmatrix} e^{-\varphi/2} & e^{\varphi/2} \chi \\ 0 & e^{\varphi/2} \end{pmatrix}, \quad (1.75)$$

and the symmetric $\text{SL}(2, \mathbb{R})$ matrix \mathcal{M} constructed according to the recipe Eq. (1.61) is the usual one

$$\mathcal{M} = e^\varphi \begin{pmatrix} |\tau|^2 & \chi \\ \chi & 1 \end{pmatrix}, \quad \text{where } \tau \equiv \chi + ie^{-\varphi}, \quad (1.76)$$

is sometimes called the *axidilaton* field.

The coset representative u transforms according to the general rule $u'(x) = u(x') = gu(x)h^{-1}$ where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}), \quad \text{so } ad - bc = 1, \quad h = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2). \quad (1.77)$$

In order to preserve the upper-triangular form of the coset representative u , the compensating h transformation must be such that

$$\tan \theta = \frac{-ce^{-\varphi}}{c\chi + d}, \quad (1.78)$$

and this completely determines the transformation rules for the coordinates φ, χ : in terms of τ they take the usual fractional-linear form

$$\tau' = \frac{a\tau + b}{c\tau + d}. \quad (1.79)$$

The generators P_a, M are

$$P_{1,2} \equiv T_{1,2}, \quad M \equiv T_3. \quad (1.80)$$

The components of the left-invariant MC 1-form in the above basis are given by

$$e^1 = d\varphi, \quad e^2 = -e^\varphi d\chi, \quad \vartheta = e^2, \quad (1.81)$$

and the $\text{SL}(2, \mathbb{R})$ -invariant metric of the coset space is

$$ds^2 = g_{ab}e^a e^b = \frac{d\tau d\tau^*}{2(\Im \tau)^2}. \quad (1.82)$$

$$(\Gamma_{\text{Adj}}(u)^A_B) = \begin{pmatrix} 1 & e^\varphi \chi & -e^\varphi \chi \\ -\chi & \frac{1}{2}e^\varphi(1 - |\tau|^2) + e^{-\varphi} & -\frac{1}{2}e^\varphi(1 - |\tau|^2) \\ -\chi & -\frac{1}{2}e^\varphi(1 + |\tau|^2) + e^{-\varphi} & -\frac{1}{2}e^\varphi(1 + |\tau|^2) \end{pmatrix}. \quad (1.83)$$

The first two components of each of the three columns of the matrix

$$\left(\Gamma_{\text{Adj}}(u^{-1})^A_B \right) = \begin{pmatrix} 1 & -\chi & \chi \\ e^\varphi \chi & \frac{1}{2}e^\varphi(1 - |\tau|^2) + e^{-\varphi} & \frac{1}{2}e^\varphi(1 + |\tau|^2) - e^{-\varphi} \\ e^\varphi \chi & \frac{1}{2}e^\varphi(1 - |\tau|^2) & \frac{1}{2}e^\varphi(1 + |\tau|^2) \end{pmatrix}. \quad (1.84)$$

are the two components of each of the three Killing vectors, while the third row gives the three components of the momentum map (one for each isometry). Using the Vierbeins, and in terms of $\partial_\tau = \frac{1}{2}(\partial_\chi + ie^\varphi \partial_\varphi)$ we get the following explicit expressions for the Killing vectors, momentum map and H-compensator

$$k_1 = \tau \partial_\tau + \text{c.c.}, \quad k_2 = \frac{1}{2}(1 - \tau^2) \partial_\tau + \text{c.c.}, \quad k_3 = \frac{1}{2}(1 + \tau^2) \partial_\tau + \text{c.c.}, \quad (1.85)$$

$$(P_A) = \left(e^\varphi \chi, \frac{1}{2}e^\varphi(1 - |\tau|^2), \frac{1}{2}e^\varphi(1 + |\tau|^2) \right), \quad (1.86)$$

$$(W_A) = i\left(\frac{1}{2}, \frac{1}{2}\tau, -\frac{1}{2}\tau\right) + \text{c.c.}, \quad (1.87)$$

1.3.2 SU(1,1)/U(1)

An SU(1,1) matrix can be parametrized by two complex numbers a, b

$$u = \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}, \quad |a|^2 - |b|^2 = 1. \quad (1.88)$$

U(1) acts on these two complex numbers by multiplication $(a, b) \rightarrow e^{-i\varphi}(a, b)$. Therefore, its right action on this matrix is

$$u \longrightarrow u \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \quad (1.89)$$

It is customary to introduce the vectors V^α_\pm , $\alpha = 1, 2$ where the subindices $+, -$ refer to the U(1) weight

$$u \equiv (V^\alpha_- V^\alpha_+), \quad \Rightarrow \quad (V^\alpha_+)^* = \sigma^{1\alpha}_\beta V^\beta_-. \quad (1.90)$$

In terms of these vectors the constraint $|a|^2 - |b|^2 = 1$ takes the form

$$V^\alpha_- V^\beta_+ - V^\alpha_+ V^\beta_- = \varepsilon^{\alpha\beta}, \quad \varepsilon_{\alpha\beta} V^\alpha_- V^\beta_+ = 1. \quad (1.91)$$

This constraint can be solved by a complex number z and a real one ζ

$$\begin{cases} a = \cosh \rho + i\tilde{\xi} \frac{\sinh \rho}{\rho}, \\ b = z \cosh \rho, \end{cases} \quad \text{where } \tilde{\xi}^2 = |z|^2 \frac{\rho^2}{\tanh^2 \rho} - \rho^2. \quad (1.92)$$

In these coordinates the general $SU(1,1)$ matrix u behaves near the origin as

$$u \sim \mathbb{1}_{2 \times 2} + \tilde{\xi} i\sigma^3 + \Re(z) \sigma^1 + \Im(z) \sigma^2, \quad (1.93)$$

from which we can read the generators and find the Lie algebra

$$T_1 = \sigma^1, \quad T_2 = \sigma^2, \quad T_3 = i\sigma^3, \quad [T_A, T_B] = -2\varepsilon_{ABD} Q^{DC} T_C, \quad (1.94)$$

where $Q = \text{diag}(+, +, -)$. The Lie algebra is the same as that of $SL(2, \mathbb{R})$, with a different normalization and the metrics K_{AB}, g_{AB} are proportional. The subgroup $U(1)$ is generated by T_3 .

As we are going to see, in many instances, the adjoint index $A = 1, 2, 3$ can be replaced by a symmetric pair of indices $\alpha, \beta = 1, 2$ which can be obtained from the corresponding Pauli matrix by left multiplication with $\varepsilon_{\alpha\beta}$.

Using $U(1)$ we can always bring u to the gauge $\tilde{\xi} = 0$ in which all its components are written in terms of the unconstrained complex variable z :

$$u = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z^* \\ z & 1 \end{pmatrix}, \quad z = V_-^2 / V_-^1. \quad (1.95)$$

Then, this coset representative $u(z)$ transforms according to the general rule $u'(z) = u(z') = gu(z)h^{-1}$ with

$$g = \begin{pmatrix} u & v^* \\ v & u^* \end{pmatrix} \in SU(1,1), \quad \text{so } |u|^2 - |v|^2 = 1, \quad h = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \in U(1), \quad (1.96)$$

where

$$e^{i\varphi} = \frac{u + vz^*}{|u + vz^*|}, \quad \text{and } z' = \frac{v + u^*z}{u + v^*z}. \quad (1.97)$$

The generators P_a, M are

$$P_{1,2} \equiv T_{1,2}, \quad T_3 \equiv M. \quad (1.98)$$

Since M is diagonal and the P_a are anti-diagonal, the left-invariant 1-forms can be decomposed in terms of a complex Vielbein $e = e^1 + ie^2$ and a real connection ϑ as

$$-u^{-1}du = e^a P_a + \vartheta M = \begin{pmatrix} i\vartheta & e^* \\ e & -i\vartheta \end{pmatrix}, \quad (1.99)$$

and using $u^{-1} = \begin{pmatrix} -V^\beta + \varepsilon_{\beta\alpha} \\ +V^\beta - \varepsilon_{\beta\alpha} \end{pmatrix}$ we find that they are given by

$$\begin{aligned} e &= -\varepsilon_{\alpha\beta} V^\alpha_- dV^\beta_- = \frac{dz}{1 - |z|^2}, \\ \vartheta &= i\varepsilon_{\alpha\beta} V^\alpha_+ dV^\beta_- = \frac{i}{2} \frac{zdz^* - z^*dz}{1 - |z|^2}. \end{aligned} \quad (1.100)$$

The invariant metric is²¹

$$ds^2 = g_{ab} e^a e^b = 2 \frac{dzdz^*}{(1 - |z|^2)^2}. \quad (1.102)$$

The Killing vectors and momentum maps can be computed by using the general formulae Eqs. (1.26) and (1.28). Using the definition of the adjoint action of a group on its Lie algebra Eq. (1.9):

$$T_B \Gamma_{\text{Adj}}(u^{-1})^B{}_A = u^{-1} T_A u, \Rightarrow \Gamma_{\text{Adj}}(u^{-1})^B{}_A = g^{BC} \text{Tr} \left[T_C u^{-1} T_A u \right], \quad (1.103)$$

so

$$k_A{}^a = -g^{ab} \text{Tr} \left[P_b u^{-1} T_A u \right], \quad (1.104)$$

$$P_A = \text{Tr} \left[M u^{-1} T_A u \right]. \quad (1.105)$$

It is convenient to use the symmetric pairs $\alpha\beta$ to label the Killing vectors and momentum maps. We get, introducing a global factor of $-i$ to make them real

$$k^{(\alpha\beta)1} + i k^{(\alpha\beta)2} = i V^\alpha_- V^\beta_-, \quad (1.106)$$

$$P^{\alpha\beta} = 2 V^{(\alpha}_+ V^{\beta)}_-, \quad (1.107)$$

and

$$k_A{}^a = -i k^{(\alpha\beta)a} \varepsilon_{\alpha\gamma} T_A{}^\gamma{}_\beta, \quad P_A = -i P^{(\alpha\beta)} \varepsilon_{\alpha\gamma} T_A{}^\gamma{}_\beta. \quad (1.108)$$

²¹The change of variables

$$-i\tau = \frac{1-z}{1+z}, \quad (1.101)$$

brings the metric of the $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ coset Eq. (1.82) to this one.

1.3.3 $E_{7(+7)}/SU(8)$

The $E_{7(+7)}/SU(8)$ coset representative can be written in the convenient form of a $Usp(28, 28)$ matrix

$$U = \begin{pmatrix} u^{AB}{}_{IJ} & v^{*AB}{}^{IJ} \\ v_{AB}{}_{IJ} & u_{AB}^{*IJ} \end{pmatrix}, \quad \begin{aligned} U^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} U &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \\ U^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} U &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \end{aligned} \quad (1.109)$$

where all the indices are complex $SU(8)$ indices raised and lowered by complex conjugation that occur in antisymmetric pairs AB, IJ so that

$$U^{-1} = \begin{pmatrix} (u^\dagger)^{IJ}{}_{AB} & -(v^\dagger)^{IJ}{}^{AB} \\ -(v^T)_{IJ}{}_{AB} & (u^T)_{IJ}{}^{AB} \end{pmatrix}. \quad (1.110)$$

The $Usp(28, 28)$ condition implies for the 28×28 matrices u and v the two conditions

$$\begin{aligned} u^\dagger u - v^\dagger v &= \mathbb{1}, \\ v^T u + u^T v &= 0. \end{aligned} \quad (1.111)$$

$E_{7(+7)}$ acts on the AB indices and the compensating (scalar-dependent) $SU(8)$ transformations act on the IJ indices. A parametrization in terms of independent scalar fields can be found, for instance in Ref. [40].

Each column of the above matrix provides a set of complex vectors labeled by the pair IJ transforming in the fundamental (i.e **56**) representation under $E_{7(+7)}$. The action of this group on the fundamental representation in this complex basis can be described as follows: consider, for instance the complex combinations of the electric and magnetic 2-form field strengths \mathcal{F}^{AB} given by

$$\mathcal{F}^{AB} \equiv \frac{1}{4\sqrt{2}} (F^{ij} - iG_{ij}) \Gamma^{ij}{}_{AB}, \quad (1.112)$$

where ij are antisymmetric pairs of real $SL(8)$ indices, the F^{ij} are the 28 electric field strengths of the theory, the G_{ij} are the 28 magnetic field strengths defined from the Lagrangian of the theory L by

$$G_{ij\mu\nu} \equiv \frac{1}{4} \star \frac{\partial L}{\partial F^{ij}{}_{\mu\nu}}, \quad (1.113)$$

and the Γ^{ij} s are the $SO(8)$ gamma matrices. Then, the infinitesimal action of $E_{7(+7)}$ on the fundamental representation is given by

$$\delta \begin{pmatrix} \mathcal{F}^{AB} \\ \mathcal{F}_{AB}^* \end{pmatrix} = \begin{pmatrix} 2\Lambda^A{}_{[C}\delta^B]{}_D & \Sigma^{ABCD} \\ \Sigma^*{}_{ABCD} & -2\Lambda^C{}_{[A}\delta^D]{}_B \end{pmatrix} \begin{pmatrix} \mathcal{F}^{CD} \\ \mathcal{F}_{CD}^* \end{pmatrix}, \quad (1.114)$$

where the Λ^A_B are the anti-Hermitian parameters of infinitesimal $SU(8)$ transformations, (i.e. $\Lambda^{*A}_B = -\Lambda^B_A$ and $\Lambda^A_A = 0$) and where the off-diagonal infinitesimal parameters Σ_{ABCD} are complex self-dual, that is

$$(\Sigma^{ABCD})^* \equiv \Sigma^*_{ABCD} = \frac{1}{4!} \varepsilon_{ABCDEFGH} \Sigma^{EFGH}. \quad (1.115)$$

The generators of $E_{7(+7)}$ in this representation are, therefore,

$$\begin{pmatrix} 2\Lambda^{[A}_{[C}\delta^{B]}_{D]} & \Sigma^{ABCD} \\ \Sigma^*_{ABCD} & -2\Lambda^{[C}_{[A}\delta^{D]}_{B]} \end{pmatrix} = \Lambda^E_F \mathcal{T}_E + \Sigma^{EFGH} \mathcal{T}_{EFGH} \quad (1.116)$$

where

$$\mathcal{T}_E = \begin{pmatrix} T_E^{AB}_{CD} & 0 \\ 0 & -T_E^{CD}_{AB} \end{pmatrix}, \quad \mathcal{T}_{EFGH} = \begin{pmatrix} 0 & T_{EFGH}^{ABCD} \\ T_{EFGH}^{ABCD} & 0 \end{pmatrix}, \quad (1.117)$$

where, in its turn,

$$\begin{aligned} T_E^{AB}_{CD} &= 2 \left(\delta^{[A}_E \delta^{B]}_{CD} - \frac{1}{8} \delta^E_E \delta^{AB}_{CD} \right), \\ T_{EFGH}^{ABCD} &= \delta_{EFGH}^{ABCD}, \\ T_{EFGH}^{ABCD} &= \frac{1}{4!} \varepsilon_{EFGHABCD}. \end{aligned} \quad (1.118)$$

The generators \mathcal{T}_E of $\mathfrak{h} = \mathfrak{su}(8)$ will also be denoted M_E and the generators of the complement \mathcal{T}_{EFGH} will be denoted by P_{EFGH} . In order to avoid confusion, in this section the indices in the adjoint representation of $E_{7(+7)}$ will be $\mathbf{A}, \mathbf{B}, \dots$ and correspond to the pairs $\frac{E}{F}$ plus the quartets $EFGH$. The metric in $\mathfrak{e}_{7(+7)}$ $g_{\mathbf{AB}}$ will be

$$g_{\mathbf{AB}} \equiv \text{Tr}(\mathcal{T}_A \mathcal{T}_B). \quad (1.119)$$

Using the explicit form of the generators \mathcal{T}_A written above one can also compute explicitly the structure constants $f_{\mathbf{AB}}^{\mathbf{C}}$, the Killing metric $K_{\mathbf{AB}}$ and the metric $g_{\mathbf{AB}}$. The latter's and its inverse's only non-vanishing components are

$$\begin{aligned} g_{\frac{E}{F} \frac{G}{H}} &= 12(\delta^E_H \delta^G_F - \frac{1}{8} \delta^E_F \delta^G_H), & g_{ABCD EFGH} &= \frac{1}{12} \varepsilon_{ABDCEFGH}, \\ g_{\frac{E}{F} \frac{G}{H}} &= \frac{1}{12}(\delta^{EG} \delta_{FH} - \frac{1}{8} \delta^E_F \delta^G_H), & g^{ABCD EFGH} &= \frac{1}{2 \cdot 4!} \varepsilon^{ABDCEFGH}. \end{aligned} \quad (1.120)$$

The Vielbein and H-connection are defined by

$$-U^{-1}dU = e^{IJKL}P_{IJKL} + \vartheta^I{}_J M^J{}_I = \begin{pmatrix} \vartheta^{IJ}{}_{KL} & e^{IJKL} \\ e^*_{IJKL} & -\vartheta^{KL}{}_{IJ} \end{pmatrix}, \quad (1.121)$$

and, using Eq. (1.109) one finds that they are given by

$$\begin{aligned} \vartheta^{IJ}{}_{KL} &= -(u^\dagger)^{IJ}{}_{AB} du^{AB}{}_{KL} + (v^\dagger)^{IJ}{}_{AB} dv_{AB}{}_{KL}, \\ e^{IJKL} &= -(u^\dagger)^{IJ}{}_{AB} dv^{*AB}{}_{KL} + (v^\dagger)^{IJ}{}_{AB} du^{*AB}{}_{KL}. \end{aligned} \quad (1.122)$$

From the Maurer-Cartan equations it follows that

$$\mathcal{D}e^{IJKL} \equiv de^{IJKL} - \vartheta^{IJ}{}_{MN} \wedge e^{MNKL} + e^{IJMN} \wedge \vartheta^*_{MN}{}^{KL} = 0, \quad (1.123)$$

$$R^{IJ}{}_{KL} \equiv d\vartheta^{IJ}{}_{KL} - \vartheta^{IJ}{}_{mn} \wedge \vartheta^{mn}{}_{KL} = e^{IJMN} \wedge e^*_{MNKL}. \quad (1.124)$$

Again, in order to compute the Killing vectors and momentum maps we use the same reasoning as in the previous example, arriving to

$$\begin{aligned} k_{\mathbf{A}}^{EFGH} &= -g^{EFGHABCD} \text{Tr} [\mathcal{T}_{ABCD} U^{-1} \mathcal{T}_{\mathbf{A}} U], \\ P_{\mathbf{A}}^E{}_F &= g^E{}_F{}^G{}_H \text{Tr} \left[\mathcal{T}_G{}^H U^{-1} \mathcal{T}_{\mathbf{A}} U \right]. \end{aligned} \quad (1.125)$$

2 Noether 1-forms and dualization

For each isometry of the metric $\mathcal{G}_{mn}(\phi)$ with Killing vector $k_A{}^m(\phi)$, the σ -model action Eq. (1.20) has a global symmetry with $\delta_A \phi^m = k_A{}^m(\phi)$. According to Noether's first theorem, there is a current density $j_A{}^\mu$ associated to each of them which is conserved on-shell:

$$j_A{}^\mu = \sqrt{|g|} g^{\mu\nu} \mathcal{G}_{mn} k_A{}^m \partial_\nu \phi^n, \quad \partial_\mu j_A{}^\mu = -k_A{}^m \frac{\delta S}{\delta \phi^m}. \quad (2.1)$$

The Noether 1-form is defined by

$$j_A = \mathcal{G}_{mn} k_A{}^m d\phi^n, \quad (2.2)$$

and, for the choice of metric Eq. (1.24), and using the expression Eq. (1.26), it can be written in the form

$$j_A = \text{Tr} [T_A \mathcal{D} u u^{-1}] = -g_{AB} \Gamma_{\text{Adj}}(u)^B{}_a e^a, \quad (2.3)$$

which can also be obtained from the explicitly right-invariant expression Eq. (1.59) for the transformation $\delta_A u = T_A u$.²²

The above expression makes it easier to show that not all of these 1-forms are independent: they satisfy dim H scalar-dependent relations:

$$j_A \Gamma_{\text{Adj}}(u)^A{}_i = \text{Tr} \left[M_i u^{-1} \mathcal{D}u \right] = 0, \quad (2.5)$$

where we have used Eq. (1.58) and the orthogonality of the basis of \mathfrak{h} and \mathfrak{k} in symmetric spaces. Observe that the above expression *is not simply* $j_i = 0$.

For the rest of the components we get the non-vanishing expression

$$j_A \Gamma_{\text{Adj}}(u)^A{}_a = \text{Tr} \left[P_a u^{-1} \mathcal{D}u \right] = -g_{ab} e^b. \quad (2.6)$$

This expression and the previous one appear in the equations of motion of the scalars Eq. (1.71) and in the gauge identities Eq. (1.70). Thus, these can be rewritten, respectively in the form

$$e_m{}^a \mathcal{D} \star \left[j_A \Gamma_{\text{Adj}}(u)^A{}_a \right] = 0, \quad (2.7)$$

$$\mathcal{D} \star \left[j_A \Gamma_{\text{Adj}}(u)^A{}_i \right] = 0. \quad (2.8)$$

Using the explicit form of the H-covariant derivatives, we find that these equations can be rewritten in the following form, much more directly related to the conservation of the Noether 1-forms:

$$e_m{}^a \Gamma_{\text{Adj}}(u)^A{}_a d \star j_A = 0, \quad (2.9)$$

$$\Gamma_{\text{Adj}}(u)^A{}_i d \star j_A = 0. \quad (2.10)$$

Combining these equations with the explicit form of the Killing vectors Eq. (1.26) we can easily prove the off-shell relation

$$k_A{}^m \frac{\delta S_{\text{Gauged}}}{\delta \phi^m} = -d \star j_A. \quad (2.11)$$

The moral of these results is that the equations of motion of the scalars can be seen as combinations (projections) of some more fundamental equations: the conservation

²²The complete infinitesimal transformation of u must include the compensating H-transformations that act from the right:

$$\delta_A u = T_A u + u W_A{}^i M_i, \quad (2.4)$$

but they can be safely ignored in the right-invariant expression.

laws of the Noether currents. The latter can completely replace the former. But only the Noether 1-forms can be dualized.

The Noether 1-forms are closed on-shell and, on-shell, they can be dualized by introducing as many $(d - 2)$ -forms B_A related to them by

$$\star j_A \equiv dB_A \equiv H_A, \quad (2.12)$$

solving locally the conservation laws. As usual, the Bianchi identities of the original fields become the equations of motion of the dual ones. The obvious candidate to Bianchi identity is

$$\mathcal{D} \left(\Gamma_{\text{Adj}}(u^{-1})^a{}_A g^{AB} j_B \right) = 0, \quad (2.13)$$

by virtue of Eq. (2.6) and Cartan structure equation $\mathcal{D}e^a = 0$ which is equivalent to the first set of Maurer–Cartan Eqs. (1.14). Then, the equations of motion satisfied by the 2-forms are²³

$$\mathcal{D} \left(\Gamma_{\text{Adj}}(u)^A{}_a \star H_A \right) = 0. \quad (2.14)$$

There are only $\dim G - \dim H$ equations, which means that we cannot solve for all the H_A . We must also use the constraint

$$\Gamma_{\text{Adj}}(u)^A{}_i H_A = 0, \quad (2.15)$$

which follows from Eq. (2.5).

It would be desirable to have a kinetic term for the $(d - 2)$ -forms B_A from which the equations of motion (2.14) could be derived. The simplest candidate would be

$$S = \int \frac{1}{2} \mathfrak{M}^{AB} H_A \wedge \star H_B, \quad (2.16)$$

where the scalar-dependent matrix \mathfrak{M}^{AB} is defined by

$$\mathfrak{M}^{AB} \equiv \Gamma_{\text{Adj}}(u)^A{}_a \Gamma_{\text{Adj}}(u^{-1})^a{}_C g^{CB}, \quad (2.17)$$

and is obviously singular, with $\text{rank}(\mathfrak{M}^{AB}) = \dim G - \dim H$ because $\mathfrak{M}^{AB} \Gamma_{\text{Adj}}(u^{-1})^i{}_A = 0 \forall i$. This means that the combinations $\Gamma_{\text{Adj}}(u)^A{}_i H_A$ which are constrained to vanish, do not enter the action. Observe that \mathfrak{M}^{AB} transforms under global G transformations according to

$$\mathfrak{M}'^{AB} = \Gamma_{\text{Adj}}(g)^A{}_{A'} \Gamma_{\text{Adj}}(g)^B{}_{B'} \mathfrak{M}^{A'B'}, \quad (2.18)$$

so the above action is invariant.

Observe also that the σ -model action can be written in terms of the same singular matrix as

²³We use $\Gamma_{\text{Adj}}(u)^A{}_a = g_{ab} \Gamma_{\text{Adj}}(u^{-1})^b{}_B g^{BA}$.

$$S = \int \frac{1}{2} \mathfrak{M}^{AB} j_A \wedge \star j_B. \quad (2.19)$$

Furthermore, observe that, if we define

$$\Gamma_{\text{Adj}}(u)^A{}_a B_A \equiv \mathcal{B}_a, \quad \Gamma_{\text{Adj}}(u)^A{}_i B_A \equiv \mathcal{B}_i, \quad (2.20)$$

and we impose the restriction $\mathcal{B}_i = 0$, the above action can be rewritten in the form

$$S = \int \frac{1}{2} g^{ab} \mathcal{H}_a \wedge \star \mathcal{H}_b, \quad \text{where } \mathcal{H}_a = \mathcal{D} \mathcal{B}_a. \quad (2.21)$$

Notice, however, that the restriction $\mathcal{B}_i = 0$ is not equivalent to Eq. (2.15).

The equations of motion that follow from the above action are simply

$$d \left(\mathfrak{M}^{AB} \star H_B \right) = 0. \quad (2.22)$$

Projecting them with $\Gamma_{\text{Adj}}(u^{-1})^a{}_A$, we get the equations of motion (2.14)

$$\Gamma_{\text{Adj}}(u^{-1})^a{}_A d \left(\mathfrak{M}^{AB} \star H_B \right) = g^{ab} \mathcal{D} \left(\Gamma_{\text{Adj}}(u)^A{}_b \star H_A \right) = 0, \quad (2.23)$$

whose solution is

$$g^{ab} \Gamma_{\text{Adj}}(u)^A{}_b \star H_A \propto e^a. \quad (2.24)$$

However, if we project the equations of motion with $\Gamma_{\text{Adj}}(u^{-1})^i{}_A$ we get a non-trivial constraint:

$$\Gamma_{\text{Adj}}(u^{-1})^i{}_A d \left(\mathfrak{M}^{AB} \star H_B \right) = -e^a f_{ab}{}^i g^{bc} \Gamma_{\text{Adj}}(u)^A{}_c \star H_A = 0, \quad (2.25)$$

which, upon use of the previous solutions gives a non-trivial constraint that we do not want:

$$R(\vartheta)^i = 0. \quad (2.26)$$

We have not found any completely satisfactory way of solving this problem in general.

Notice that a similar problem appeared with dualization of 3-form potential A_3 of eleven dimensional supergravity. Its action [41] contains, besides the kinetic term of A_3 and interaction of A_3 with fermions, the Chern-Simons type term $dA_3 \wedge A_3 \wedge A_3$, and this makes impossible to construct a dual action including 6-form potential A_6 instead of A_3 [42, 43].

However, there exists the duality invariant action of 11-d supergravity including both A_6 and A_3 potentials [8]. It was constructed using the PST (Pati-Sorokin-Tonin) approach [11, 12] and reproduce a (nonlinear) duality relation between the (generalized) field strengths of A_6 and A_3 as a gauge fixed version of the equations of motion.

Notice that this action can be presented formally as a sigma model action [10] for a supergroup with fermionic generator associated to A_3 and bosonic generators associated to A_6 [44].

Then it is natural to expect that the similar situation occurs in our case of dualization of “non-Abelian” scalars. Even if the non-existence of a consistent way to write dual action in terms of only $(d - 2)$ forms dual to a scalars parametrizing a non-Abelian coset were proved, this would not prohibit the existence of a PST-type action involving both the scalars and the $(d - 2)$ forms and producing the duality equations (2.12) as a gauge fixed version of the equations of motion. Moreover, for the particular case of $SU(1, 1)/U(1)$ coset such action was constructed in [9], where it was also incorporated in the complete action of type IIB supergravity.

The generalization of the action from Ref. [9] for the generic case of scalars parametrizing a symmetric space G/H reads

$$S_{\text{PST}} = \int L_{\text{PST}}, \quad (2.27)$$

$$L_{\text{PST}} = \frac{1}{2} g^{AB} j_A \wedge \star j_B + \frac{1}{2} g_{ij} F^i \wedge \star F_j + \frac{(-1)^d}{2} g^{AB} \mathcal{H}_A \wedge v i_v \star \mathcal{H}_B, \quad (2.28)$$

where

$$\begin{aligned} \mathcal{H}_A &= H_A + \star j_A + \Gamma_{\text{Adj}}(u^{-1})^i_A \star F_i \\ &\equiv H_A + \star j_A + P^i_A \star F_i, \end{aligned} \quad (2.29)$$

the one-form v is constructed from the PST scalar $a(x)$, the (would be) auxiliary field of the PST formalism,

$$v = \frac{da(x)}{\sqrt{\partial a \partial a}}, \quad v_\mu = \frac{\partial_\mu a(x)}{\sqrt{\partial a \partial a}}, \quad (2.30)$$

and $F_i = dx^\mu F_{\mu i}(x)$ is an auxiliary one-form carrying the index of H -generators. The contraction symbol is defined, as usual, by

$$i_v j_A = v^\mu j_{\mu A}, \quad i_v H_A = \frac{1}{(d-2)!} dx^{v(D-2)} \wedge \dots \wedge dx^{v_1} H_{v_1 \dots v_{(d-2)} \mu} v^\mu. \quad (2.31)$$

We assume the derivative of the PST scalar to be a time-like vector so that the square root in denominator is well defined (in our mostly minus signature), $v_\mu v^\mu = 1$, and

$$H_A = v \wedge i_v H_A + \star(v \wedge i_v \star H_A), \quad (2.32)$$

is valid for any $(d - 2)$ -form and, in particular, for our $H_A = dB_A$.

The study of this action and derivation of the duality conditions from its equations of motion is out of the scope of this paper. Here we would like to stress the rôle the

momentum map (1.28) plays in it: the auxiliary one-form F_i always enter the action in contraction $F_i P^i_A$. Indeed, this is the case for \mathcal{H}_A (2.29), and, due to $j_A \Gamma_{\text{Adj}}(u)^A_i = 0$ (2.5), the first two terms of the Lagrangian (2.28) can also be collected in $\frac{1}{2}g^{AB}(j_A + P^i_A F_i) \wedge \star(j_B + P^j_B F_j)$,

$$L_{\text{PST}} = \frac{1}{2}g^{AB}(j_A + P^i_A F_i) \wedge \star(j_B + P^j_B F_j) + \frac{(-1)^d}{2}g^{AB} \mathcal{H}_A \wedge v i_v \star \mathcal{H}_B. \quad (2.33)$$

We hope to return to the study the properties of this action and its applications in supergravity context in future publications.

2.1 Examples

2.1.1 $\text{SL}(2, \mathbb{R})/\text{SO}(2)$

A short calculation gives

$$\mathcal{D}uu^{-1} = \frac{1}{2}d\mathcal{M}\mathcal{M}^{-1} = j_A B^{AB} T_B, \quad (2.34)$$

where the Noether 1-forms are given by

$$\begin{aligned} j_1 &= \frac{1}{4}e^{2\varphi}d|\tau|^2, \\ j_2 &= -\frac{1}{4}e^{2\varphi}\chi d|\tau|^2 + \frac{1}{4}e^{2\varphi}(1 + |\tau|^2)d\chi, \\ j_3 &= +\frac{1}{4}e^{2\varphi}\chi d|\tau|^2 + \frac{1}{4}e^{2\varphi}(1 - |\tau|^2)d\chi. \end{aligned} \quad (2.35)$$

As expected, they are not independent: they are related by one (dim H) relation of the form Eq. (2.5) where $\Gamma_{\text{Adj}}(u)^A_3$ is the third column of the $\text{SO}(2, 1)$ matrix in Eq. (1.83).

The singular matrix \mathfrak{M}^{AB} in the action Eq. (2.16) is

$$\left(\mathfrak{M}^{AB}\right) = e^{2\varphi} \begin{pmatrix} |\tau|^2 & \frac{1}{2}(1 - |\tau|^2)\chi & -\frac{1}{2}(1 + |\tau|^2)\chi \\ \frac{1}{2}(1 - |\tau|^2)\chi & e^{-2\varphi} + \frac{1}{4}(1 - |\tau|^2)^2 & -\frac{1}{4}(1 - |\tau|^4) \\ -\frac{1}{2}(1 + |\tau|^2)\chi & -\frac{1}{4}(1 - |\tau|^4) & -e^{-2\varphi} + \frac{1}{4}(1 + |\tau|^2)^2 \end{pmatrix}. \quad (2.36)$$

The combinations of dual $(d - 1)$ -form field strengths H_A that occur in that action are

$$\begin{aligned} \Gamma_{\text{Adj}}(u)^A_1 H_A &= H_1 - \chi(H_2 + H_3), \\ \Gamma_{\text{Adj}}(u)^A_2 H_A &= e^\varphi \chi [H_1 - \frac{1}{2}\chi(H_2 + H_3)] + \frac{1}{2}e^{-\varphi}(H_2 + H_3) + \frac{1}{2}e^{-\varphi}(H_2 - H_3), \end{aligned} \quad (2.37)$$

and the constraint Eq. (2.15) is

$$\Gamma_{\text{Adj}}(u)^A{}_3 H_A = e^\varphi \chi H_1 + \frac{1}{2} e^\varphi (H_2 + H_3) - \frac{1}{2} e^\varphi |\tau|^2 (H_2 - H_3) = 0. \quad (2.38)$$

The relation between these three $(d-1)$ -form field strengths and the scalars (the pullbacks of the Vierbeins) is

$$\Gamma_{\text{Adj}}(u)^A{}_a H_A = \star e^a, \quad (2.39)$$

and, with the help of the above constraint we can invert it, expressing entirely the three field strengths in terms of the scalars. The relations are equivalent to $H_A = \star j_A$.

2.1.2 $E_{7(+7)}/\text{SU}(8)$

Using the same properties we used to find the Killing vectors and momentum maps Eqs. (1.125) we find the Noether 1-forms are given by $j_{\mathbf{A} ABCD} e^{ABCD}$ where the e^{ABCD} are the Vielbein and the components are given by

$$j_{\mathbf{A} ABCD} = \text{Tr} \left[\mathcal{T}_{ABCD} U^{-1} \mathcal{T}_{\mathbf{A}} U \right]. \quad (2.40)$$

The explicit expressions can be easily computed using the generators and (inverse) coset representatives given above. For instance,

$$\begin{aligned} j_{\mathbf{E}}^E{}_{ABCD} &= \frac{1}{12} \varepsilon_{ABCDG H M N} \left[(v^\dagger)^{GH F J} u_{E J}^{* M N} + (u^\dagger)^{GH}{}_{E J} v^{* F J M N} \right] \\ &\quad - 2 \delta_{ABCD}{}^{G H M N} \left[(v^T)_{G H E J} u^{F J}{}_{M N} + (u^T)_{G H}{}^{F J} v_{E J M N} \right]. \end{aligned} \quad (2.41)$$

Observe that the 1-forms $j_{\mathbf{E}}^E{}_{ABCD} e^{ABCD}$ are purely imaginary.

3 NGZ 1-forms and dualization in $d = 4$

The bosonic action of all 4-dimensional ungauged supergravities (and many other interesting theories as well) is of the generic form

$$\begin{aligned} S[g_{\mu\nu}, A^\Lambda{}_\mu, \phi^m] &= \int d^4x \sqrt{|g|} \left\{ R + \mathcal{G}_{mn}(\phi) \partial_\mu \phi^m \partial^\mu \phi^n \right. \\ &\quad \left. + 2 \Im \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^\Sigma{}_{\mu\nu} - 2 \Re \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^\Sigma{}_{\mu\nu} \right\}, \end{aligned} \quad (3.1)$$

where the indices $\Lambda, \Sigma = 1, \dots, \bar{n}$ (the total number of fundamental vector fields) and where $\mathcal{N}_{\Lambda\Sigma}(\phi)$ is known as the period matrix and it is symmetric and, by convention, has a negative definite imaginary part. The σ -model metric $\mathcal{G}_{mn}(\phi)$ is that of a Riemannian symmetric space in all $\mathcal{N} > 2$ cases (and in many other cases as well) and

this is the case that we want to consider here in order to apply the results derived in the previous sections.

First of all, we want to rewrite this action in differential-form language and using the coset representative u :

$$S = \int \left\{ -\star R + \text{Tr}[u^{-1}\mathcal{D}u \wedge \star u^{-1}\mathcal{D}u] - 4\Im\mathcal{N}_{\Lambda\Sigma}F^\Lambda \wedge \star F^\Sigma - 4\Re\mathcal{N}_{\Lambda\Sigma}F^\Lambda \wedge F^\Sigma \right\}. \quad (3.2)$$

Then, we define the dual vector field strengths

$$G_\Lambda \equiv \Im\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma + \Re\mathcal{N}_{\Lambda\Sigma}F^\Sigma, \quad \text{or} \quad G_\Lambda^+ \equiv \mathcal{N}_{\Lambda\Sigma}^*F^\Sigma. \quad (3.3)$$

The last relation is known as a *(linear) twisted self-duality constraint*. Defining the symplectic vector of vector field strengths

$$(\mathcal{F}^M) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \quad (3.4)$$

the equations of motion and the Bianchi identities can be written together as

$$d\mathcal{F}^M = 0, \quad (3.5)$$

which can be solved locally by assuming the existence of 1-form potentials \mathcal{A}^M

$$\mathcal{F}^M = d\mathcal{A}^M. \quad (3.6)$$

This set of equations is invariant under linear transformations

$$\mathcal{F}'^M = S^M_N \mathcal{F}^N, \quad S \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (3.7)$$

As it is well known, the preservation of the twisted self-duality constraint requires the simultaneous transformation of the period matrix according to the rule

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}. \quad (3.8)$$

The preservation of the symmetry of \mathcal{N} and of the negative definiteness of $\Im\mathcal{N}$ (together with the preservation of the energy-momentum tensor) require S to be an $\text{Sp}(2\bar{n}, \mathbb{R})$ transformation, that is

$$S^T \Omega S = \Omega, \quad \Omega \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \quad (3.9)$$

Finally, if these transformations are going to be symmetries of the equations of motion, the form of the period matrix as a function of the scalars must be preserved, and this requires the above transformation rule for \mathcal{N} to be equivalent to a transformation of the scalars:

$$\mathcal{N}'(\phi) = \mathcal{N}(\phi'). \quad (3.10)$$

This transformation of the scalars must be an isometry of the σ -model metric. Thus, the symmetries of the equations of motion of the theory are the group G of the isometries of the σ -model metric which act on the vector fields embedded in the symplectic group²⁴. These isometries always leave invariant the scalars' kinetic term but only some of them may leave invariant the whole action because many involve electric-magnetic duality rotations. Thus, there is a Noether 1-form for each isometry of the scalar sector, and we are going to denote it by a σ index,

$$j_A^{(\sigma)} = 2\text{Tr} \left[T_A \mathcal{D} u u^{-1} \right], \quad (3.11)$$

but, in general, they do not have a standard completion to Noether 1-forms of the full theory. A completion does, nevertheless, exist in all cases and it was found by Gaillard and Zumino in Ref. [16]. To construct the Noether–Gaillard–Zumino (NGZ) 1-form we first need to define the infinitesimal generators of G in the representation in which they act on the vector fields, $\{\mathcal{T}_A\}$. By assumption $\mathcal{T}_A \in \mathfrak{sp}(2\bar{n}, \mathbb{R})$ and

$$\delta_A \mathcal{F}^M = \mathcal{T}_A^M{}_N \mathcal{F}^N. \quad (3.12)$$

Then, the NGZ 1-forms are given by

$$j_A = j_A^{(\sigma)} - 2\mathcal{T}_A^M{}_N \star (\mathcal{F}^N \wedge \mathcal{A}_M), \quad \mathcal{A}_M = \Omega_{MN} \mathcal{A}^N. \quad (3.13)$$

Observe that the NGZ 1-forms are not invariant under the gauge transformations of the 1-forms $\delta_\sigma \mathcal{A}^M = d\sigma^M$ precisely because the Noether 1-forms $j_A^{(\sigma)}$ are. Let us check that they are conserved on-shell. First, the conservation equation takes the form

$$d \star j_A = d \star j_A^{(\sigma)} - 2\mathcal{T}_A^M{}_N \mathcal{F}^N \wedge \mathcal{F}_M, \quad (3.14)$$

where we have used Eqs. (3.5). Now we are going to see that this equation is proportional to the projection of the scalar equations of motion with the Killing vectors. The scalar equations of motion are

$$\frac{\delta S}{\delta \phi^m} = 2 \frac{\delta S_{\text{Gauged}}}{\delta \phi^m} - 4F^\Lambda \frac{\partial G_\Lambda}{\partial \phi^m} = 0, \quad (3.15)$$

where S_{Gauged} is the σ -model action normalized as in Eq. (1.65). Contracting with the Killing vectors we get

$$k_A^m \frac{\delta S}{\delta \phi^m} = -2 \left[d \star j_A^{(\sigma)} + 2F^\Lambda k_A^m \frac{\partial G_\Lambda}{\partial \phi^m} \right] = 0, \quad (3.16)$$

²⁴We are going to ignore the possibility of scalars which do not couple to the vector fields because this simply does not happen in $\mathcal{N} > 2$ supergravities.

where we have used Eq. (2.11). The infinitesimal transformation rule for the period matrix follows from Eq. (3.10) and Eq. (3.8)²⁵

$$k_A{}^m \frac{\partial \mathcal{N}_{\Lambda\Sigma}}{\partial \phi^m} = \mathcal{T}_{A\Lambda\Sigma} - \mathcal{N}_{\Lambda\Omega} \mathcal{T}_A{}^\Omega{}_\Sigma + \mathcal{T}_{A\Lambda}{}^\Omega \mathcal{N}_{\Omega\Sigma} - \mathcal{N}_\Lambda \mathcal{T}_A{}^{\Omega\Delta} \mathcal{N}_{\Delta\Sigma}, \quad (3.19)$$

and, replacing it in Eq. (3.16), we find the conservation equations as combinations of the equations of motion:

$$k_A{}^m \frac{\delta S}{\delta \phi^m} = -2 \left[d \star j_A^{(\sigma)} - 2 \mathcal{T}_A{}^M{}_N \mathcal{F}_M \wedge \mathcal{F}^N \right] = 0. \quad (3.20)$$

The conservation of the NGZ 1-forms can be solved locally by the introduction of 2-forms B_A such that

$$\star j_A = dB_A, \quad \Rightarrow \quad \star j_A^{(\sigma)} = dB_A + 2 \mathcal{T}_A{}^M{}_N \mathcal{F}^N \wedge \mathcal{A}_M \equiv H_A, \quad (3.21)$$

where H_A are the 3-form field strengths, gauge invariant under

$$\delta_\sigma \mathcal{A}^M = d\sigma^M, \quad \delta_\sigma B_A = d\sigma_A - 2 \mathcal{T}_A{}^M{}_N \mathcal{F}^N \wedge d\sigma_M, \quad (3.22)$$

and satisfying the Bianchi identities

$$dH_A - 2 \mathcal{T}_{AMN} \mathcal{F}^M \wedge \mathcal{F}^N = 0. \quad (3.23)$$

The equations of motion have the same form as in the general case studied in Section 2.

Observe that the NGZ currents are subject to the same constraint as the Noether currents, Eq. (2.5), because $k_A{}^m \Gamma_{\text{Adj}}(u)^A{}_i = 0$ and because of Eq. (3.20).²⁶ Together, they lead to the constraints

$$\mathcal{T}_{iMN} \Gamma(u^{-1})^M{}_P \Gamma(u^{-1})^N{}_Q \mathcal{F}^P \wedge \mathcal{F}^Q = 0, \quad \text{and} \quad H_A \Gamma_{\text{Adj}}(u)^A{}_i. \quad (3.24)$$

²⁵We use the following infinitesimal form for the symplectic matrix S :

$$S^M{}_N \sim \delta^M{}_N + \sigma^A \mathcal{T}_A{}^M{}_N, \quad \text{where} \quad \mathcal{T}_A{}^P{}_{[M} \Omega_{N]P} = 0. \quad (3.17)$$

The different block components of $\mathcal{T}_A{}^M{}_N$ are defined by

$$\left(\mathcal{T}_A{}^M{}_N \right) = \begin{pmatrix} \mathcal{T}_A{}^{\Lambda\Sigma} & \mathcal{T}_A{}^{\Lambda\Sigma} \\ \mathcal{T}_{A\Lambda\Sigma} & \mathcal{T}_{A\Lambda}{}^\Sigma \end{pmatrix}, \quad \text{with} \quad \mathcal{T}_A{}^{\Lambda\Sigma} = -\mathcal{T}_{A\Sigma}{}^\Lambda, \quad \mathcal{T}_{A\Lambda\Sigma} = \mathcal{T}_{A\Sigma\Lambda}, \quad \mathcal{T}_A{}^{\Lambda\Sigma} = \mathcal{T}_A{}^{\Sigma\Lambda}. \quad (3.18)$$

²⁶Otherwise, the NGZ currents would represent too many degrees of freedom.

3.1 Supersymmetry and the momentum map

3.1.1 Supersymmetry transformations of $(d-2)$ -forms

In supersymmetric theories the $(d-2)$ -form fields dual to the scalars, B_A , must transform under supersymmetry and the algebra of the supersymmetry transformations acting on these fields, $\delta_\epsilon B_A$, must close on shell.

In the $\mathcal{N} = 1, 2$, $d = 4$ cases [45, 46] these transformations were found to have leading terms with a common structure²⁷ that can be generalized to all \mathcal{N} and, actually, to all d :

$$\begin{aligned} \delta_\epsilon B_{A\mu_1\cdots\mu_{(d-2)}} &\sim P_A^i (M_i)^I{}_J \bar{\epsilon}^J \gamma_{[\mu_1\cdots\mu_{(d-3)}} \psi_{\mu_{(d-2)}]}^I \\ &+ \mathcal{D}_m P_A^i (M_i)^I{}_J \bar{\epsilon}^J \gamma_{\mu_1\cdots\mu_{(d-2)}} \lambda^m{}_I + \cdots \end{aligned} \quad (3.25)$$

In this expression I, J are R-symmetry indices (that is, a representation of H), $\psi_{\mu I}$ are the gravitini, $\lambda^m{}_I$ are dilatini or, more generally, the supersymmetric partners of the scalars, labeled here by m, n, p , $(M_i)^I{}_J$ are the generators of the Lie algebra of H in the same representation, and the P_A^i are the momentum maps of the isometries of the coset space G/H or the holomorphic and tri-holomorphic momentum maps of Kähler-Hodge and quaternionic-Kähler spaces in $\mathcal{N} = 1, 2$, $d = 4$ theories.²⁸ Henceforth, the index A is a “global” adjoint G index. \mathcal{D} is the H -covariant derivative acting on the momentum map (or the Kähler- or $SU(2)$ -covariant derivatives in the Kähler-Hodge and quaternionic-Kähler cases and, according to the previous observation, only the i index has to be covariantized for. The additional terms in this supersymmetry transformation rule are proportional to other p -forms of the theory and are associated to the Chern-Simons terms in the $(d-1)$ -form field strengths H_A .

The second term in this proposal adopts slightly different forms depending on the theory under consideration. First of all, one can always apply the main property of the momentum map Eq. (1.46) which also appears in different guises: Eqs. (1.34) in coset spaces G/H , (A.16) and (A.19) in Kähler-Hodge spaces and, finally, (B.14) in quaternionic-Kähler spaces. Then, one can use different properties of the H -curvature so that it does not appear explicitly: Eq. (1.18) in coset spaces G/H , the Kähler-Hodge condition that identifies the Kähler 2-form \mathcal{J} with the curvature of the complex bundle in Kähler-Hodge spaces, and the condition that relates the curvature of the $SU(2)$ connection to the hyperKähler structure Eq. (B.6) in quaternionic-Kähler manifolds.

Furthermore, observe that in the second term of Eq. (B.17), the hyperini ζ^α carry a single $\text{Sp}(2n_h)$ index α but their product with the Quadbein $U_{\alpha I}{}^u$ carries an R-symmetry index I plus a hyperscalar index u , according to the general expectation.

The example in Section 4.1.3 provides additional confirmation of the universality of the above supersymmetry transformation rule.

²⁷The $\mathcal{N} = 2$ cases, Special-Kähler and Quaternionic-Kähler target spaces for the scalars, are reviewed in Appendices A.1 and B.1, respectively.

²⁸These cases are reviewed in Appendices A and B.

The above proposal looks different from the exact result obtained in superspace for the $\mathcal{N} = 8, d = 4$ theory in Ref. [47], but one has to take into account that the 2-forms and their 3-form field strengths in that reference carry “local” ($H=SU(8)$) indices instead of “global” adjoint $E_{7(+7)}$ indices. The relation between these two sets of variables is

$$\mathcal{B}_A \equiv B_B \Gamma_{\text{Adj}}(u)^B{}_A, \quad (3.26)$$

and it is not difficult to see that the supersymmetry transformation rules Eq. (3.25) split into

$$\delta_\epsilon \mathcal{B}_{i\mu_1 \dots \mu_{(d-2)}} \sim (M_i)^I{}_J \bar{\epsilon}^J \gamma_{[\mu_1 \dots \mu_{(d-3)}} \psi_{\mu_{(d-2)}]I} + \dots \quad (3.27)$$

$$\delta_\epsilon \mathcal{B}_{a\mu_1 \dots \mu_{(d-2)}} \sim f_{ab}{}^i (M_i)^I{}_J \bar{\epsilon}^J \gamma_{\mu_1 \dots \mu_{(d-2)}} e^b{}_m \lambda^m{}_I + \dots \quad (3.28)$$

In the particular case of $\mathcal{N} = 8, d = 4$ supergravity we can make use of the results of Ref. [47]. Using Weyl spinor notation and taking into account that

1. The superpartners of scalar fields $\lambda_{\underline{\alpha}I}^{ABCD}$ are split on two Weyl spinors of the form

$$\lambda_{\alpha}^{ABCD I} = \frac{2}{4!} \epsilon^{ABCD I J K L} \chi_{\alpha J K L}, \quad \text{and} \quad \bar{\lambda}_{\dot{\alpha}}^{ABCD}{}_I = -2 \delta_I^{[A} \bar{\chi}_{\dot{\alpha}}{}^{BCD]}. \quad (3.29)$$

2. The structure constants corresponding to the commutators $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{k}$, generically denoted by $f_{ab}{}^i$ in our review, take in this case the form

$$f_{AB C D E F G H}{}^I{}_J = \frac{1}{72} (\delta_{[A}^I \epsilon_{B C D] E F G H J} - \delta_{[E}^I \epsilon_{F G H] A B C D J}), \quad (3.30)$$

and the \mathfrak{h} generators and $M_i^I{}_J \mapsto T_K^I{}_J = \delta^K{}_J \delta^I{}_L - \frac{1}{8} \delta^K{}_L \delta^I{}_J$,

we find that the above generic equations, that can be extracted from the superfield results of Ref. [47], take the explicit form

$$\delta_\epsilon \mathcal{B}_{I\mu\nu} \propto T_I^K{}_L i(\epsilon_K \sigma_{[\mu} \bar{\psi}_{\nu]}^L + \psi_{K[\mu} \sigma_{\nu]}^L \bar{\epsilon}^L), \quad (3.31)$$

$$\delta_\epsilon \mathcal{B}_{I J K L \mu \nu} \propto \epsilon_{[I} \sigma_{\mu \nu} \chi_{J K L]} - \frac{1}{4!} \epsilon_{I J K L A B C D} \bar{\epsilon}^A \tilde{\sigma}_{\mu \nu} \bar{\chi}^{B C D}. \quad (3.32)$$

On the other hand, the corresponding relation for the 3-form field strengths

$$\mathcal{H}_A \equiv H_B \Gamma_{\text{Adj}}(u)^B{}_A, \quad (3.33)$$

together with Eq. (3.24) explain, from a technical point of view, why the \mathcal{H}_i were found in Ref. [47] to be dual to fermion bilinears.

3.1.2 Tensions of supersymmetric $(d-1)$ -branes

The supersymmetry transformations of the $(d-2)$ -forms into the gravitini determine the tension of the $1/2$ -supersymmetric $(d-3)$ -branes that couple to them in a κ -symmetric action (see, for instance, Ref. [45]), if any. The explicit construction of the U-duality-invariant and κ -symmetric actions of the $1/2$ -supersymmetric $(d-3)$ -branes, in the same spirit as the construction of the $SL(2, \mathbb{R})$ -invariant actions for all branes in type IIB $d=10$ supergravity in Ref. [48] or for 0-branes in $\mathcal{N}=2, 8, d=4$ supergravity in Ref. [49] has only been carried out for the $\mathcal{N}=2, d=4$ case [45]. Nevertheless, some general lessons can be learned from those results and from the general form of the supersymmetry transformations of $(d-2)$ -form potentials Eq. (3.25).

On general grounds, $(d-3)$ -branes will be characterized by charges in the adjoint representation of G , q^A and the Wess-Zumino term in their effective world-volume action will contain the leading term²⁹

$$q^A \int B_A. \quad (3.34)$$

Then, the supersymmetry transformation rule Eq. (3.25) requires the presence of a scalar-dependent factor in the kinetic term that can be identified with the tension $\mathcal{T}_{(d-3)}$:

$$\int d^{(d-3)} \xi \mathcal{T}_{(d-3)} \sqrt{|g_{(d-3)}|}, \quad (3.35)$$

which we conjecture to be of the form

$$\mathcal{T}_{(d-3)} = \sqrt{|q^A q^B P_A^i P_B^j g_{ij}|}. \quad (3.36)$$

Observe that the rank $\dim H$ matrix $P_A^i P_B^j g_{ij}$ is related to the matrix \mathfrak{M}^{AB} defined in Eq. (2.17) by

$$P_A^i P_B^j g_{ij} = g_{AB} - g_{AC} g_{BD} \mathfrak{M}^{CD}, \quad (3.37)$$

and for $(d-3)$ -brane charges in the conjugacy class $q^A q^B g_{AB} = 0$ (which is the conjugacy class of the D7- and S7-branes of $\mathcal{N}=2B, d=10$ supergravity [51])

$$\mathcal{T}_{(d-3)} = \sqrt{|q_A q_B \mathfrak{M}^{CD}|}, \quad (3.38)$$

which is the expression one would have guessed from Eq. (2.19).

Clearly, more work is needed in order to find the complete κ -invariant worldvolume actions, find the U-duality-invariant $(d-3)$ -brane tensions and, eventually, prove the above conjecture, but we think that our arguments concerning the general structure of the supersymmetry transformation Eq. (3.25) give some support to it.

²⁹T-duality-invariant Wess-Zumino terms for the κ -symmetric world-volume effective actions of all branes in maximal supergravity in any dimension have been proposed in Ref. [50].

3.1.3 Fermion shifts

The holomorphic and triholomorphic momentum maps (resp. \mathcal{P}_A and \mathcal{P}_A^x) also appear naturally in the so-called *fermion shifts* of the supersymmetry transformations of the fermions of gauged $\mathcal{N} = 1, 2, d = 4$ supergravities. For the standard gauging (using the fundamental vectors A^Λ as gauge fields for perturbative symmetries of the action), the supersymmetry transformations of the gravitini, gaugini and hyperini of $\mathcal{N} = 2, d = 4$ supergravity can be written in the form:³⁰

$$\left\{ \begin{array}{l} \delta_\epsilon \psi_{I\mu} = \mathfrak{D}_\mu \epsilon_I + \left[T^+_{\mu\nu} \epsilon_{IJ} - \frac{1}{2} S^x \eta_{\mu\nu} \epsilon_{IK} (\sigma^x)^K{}_J \right] \gamma^\nu \epsilon^J, \\ \delta_\epsilon \lambda^{Li} = i \not{D} Z^i \epsilon^L + \left[(\mathcal{G}^{i+} + W^i) \epsilon^{IJ} + \frac{i}{2} W^{ix} (\sigma^x)^L{}_K \epsilon^{KJ} \right] \epsilon_J, \\ \delta_\epsilon \zeta_\alpha = i U_{\alpha I u} \not{D} q^u \epsilon^I + N_\alpha^I \epsilon_I, \end{array} \right. \quad (3.39)$$

where the fermion shifts are given by

$$\left\{ \begin{array}{l} S^x = \frac{1}{2} g \mathcal{L}^\Lambda \mathcal{P}_\Lambda^x, \\ W^i = \frac{1}{2} g \mathcal{L}^{*\Lambda} k_\Lambda^i = -\frac{i}{2} g \mathcal{G}^{ij*} f^{*\Lambda}_{j*} \mathcal{P}_\Lambda, \\ W^{ix} = g \mathcal{G}^{ij*} f^{*\Lambda}_{j*} \mathcal{P}_\Lambda^x, \\ N_\alpha^I = g U_\alpha^I{}_u \mathcal{L}^{*\Lambda} k_\Lambda^u. \end{array} \right. \quad (3.40)$$

As usual in $\mathcal{N} > 1$, the scalar potential is given by an expression quadratic in the fermion shifts:

$$\begin{aligned} V(Z, Z^*, q) &= -6 S^{*x} S^x + 2 \mathcal{G}_{ij*} W^i W^{*j*} + \frac{1}{2} \mathcal{G}_{ij*} W^{ix} W^{*j*x} + 2 N_\alpha^I N^\alpha{}_I \\ &= g^2 \left[-\frac{1}{4} \Im \mathcal{N}^{\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma + \frac{1}{2} \mathcal{L}^{*\Lambda} \mathcal{L}^\Sigma (4 H_{uv} k_\Lambda^u k_\Sigma^v - 3 \mathcal{P}_\Lambda^x \mathcal{P}_\Sigma^x) \right. \\ &\quad \left. + \frac{1}{2} \mathcal{G}^{ij*} f^\Lambda_i f^{*\Sigma}_{j*} \mathcal{P}_\Lambda^x \mathcal{P}_\Sigma^x \right]. \end{aligned} \quad (3.41)$$

The presence of the momentum map on all those terms is due to their transformations properties under global duality transformations. To make this fact manifest and

³⁰See, for instance, Refs. [26, 27, 18]. The momentum maps carry an index Λ here which associates them to the vector field that gauges the corresponding global symmetry. It is understood in this notation that only the momentum maps associated to the gauge symmetries occur in these expressions. This notation is considerably improved by the introduction of the embedding tensor, as we are going to see.

gain more insight in the structure of these terms, it is convenient to use the *embedding tensor*³¹ ϑ_M^A . This object relates each symmetry generator (A index) to the vector field of the theory that gauges it (M index). Introducing the embedding tensor in the fermion shifts restores the (formal) symplectic invariance of the theory^{32,33}

$$\left\{ \begin{array}{l} S^x = \frac{1}{2} \mathcal{V}^M \vartheta_M^A \mathcal{P}_A^x, \\ W^i = -\frac{i}{2} \mathcal{D}^i \mathcal{V}^{*M} \vartheta_M^A \mathcal{P}_A, \\ W^{ix} = \mathcal{D}^i \mathcal{V}^{*M} \vartheta_M^A \mathcal{P}_A^x, \\ N_\alpha^I = g \mathcal{U}_\alpha^I{}_u \mathcal{V}^{*M} \vartheta_M^A \mathbf{k}_A^u, \end{array} \right. \quad (3.42)$$

while the scalar potential must take the form

$$\begin{aligned} V(Z, Z^*, q) = & -\frac{1}{4} \mathcal{M}^{MN} \vartheta_M^A \vartheta_N^B \mathcal{P}_A \mathcal{P}_B + \frac{1}{2} \mathcal{V}^{*M} \mathcal{V}^N \vartheta_M^A \vartheta_N^B (4 \mathbf{H}_{uv} \mathbf{k}_A^u \mathbf{k}_B^v - 3 \mathcal{P}_A^x \mathcal{P}_B^x) \\ & + \frac{1}{2} \mathcal{G}^{ij*} \mathcal{D}_i \mathcal{V}^M \mathcal{D}_{j*} \mathcal{V}^{*N} \vartheta_M^A \vartheta_N^B \mathcal{P}_A^x \mathcal{P}_B^x. \end{aligned} \quad (3.43)$$

Our general definition of momentum map shares the same transformation properties and, therefore, the momentum maps should occur in all the fermion shifts of all theories. The expressions given in the literature, though, are written in a different language which obscures this point. Here we are going to show in several examples how the momentum map allows one to rewrite the fermion shifts in a universal way if one makes use of the embedding tensor.

Let us consider first the $\mathcal{N} > 2, d = 4$ theories with vector multiplets (whenever possible). It is convenient to use the formulation of Ref. [57] that can describe all these theories simultaneously and in a language very close to that of the $\mathcal{N} = 2, d = 4$ theories coupled to vector multiplets.³⁴

We just need to know some details of this formulation: the $\mathcal{N} = 2$ symplectic section \mathcal{V}^M that describes the scalars in the vector multiplets and its Kähler-covariant derivative $\mathcal{D}_i \mathcal{V}^M$ are now generalized to $\mathcal{V}_{IJ}^M = -\mathcal{V}_{JI}^M$ and \mathcal{V}_i^M where the indices $I, J = 1, \dots, \mathcal{N}$ and $i, j = 1, \dots, n_V$ (the number of vector multiplets). The fermions in the supergravity multiplet are $\psi_\mu{}_I, \chi_{IJK}, \chi^{IJKLM}$ (antisymmetric in all the $SU(\mathcal{N})$ indices,

³¹The embedding tensor and its associated formalism were introduced in Refs. [52, 53, 54]. They were developed in the context of the maximal 4-dimensional supergravity in Refs. [55, 56], but its use is by no means restricted to that context (see Chapter 2 in Ref. [18] and references therein).

³²The details of such a general gauged theory have not yet been worked out in the literature.

³³The gauge coupling constant g is also replaced by the embedding tensor, since it can describe several gauge groups with different coupling constants.

³⁴So far, this formalism has been used only in ungauged supergravities. Our proposals for the fermion shifts should help to extend this formulation to the most general gauged theories.

and $\chi^{IJKLM} = \frac{1}{3!}\varepsilon^{IJKLMNO} \chi_{OPQ}$ for $\mathcal{N} = 8$).³⁵ The fermions in the generic vector supermultiplet are λ_{iI} and λ_i^{IJK} (again, antisymmetric in all the $SU(\mathcal{N})$ indices, and $\lambda_i^{IJK} = \varepsilon^{IJKL} \lambda_{iL}$ for $\mathcal{N} = 4$). There are no vector multiplets for $\mathcal{N} > 4$. However, in this formalism, several fields of the $\mathcal{N} = 6$ theory are treated as if belonging to a vector supermultiplet and one has two additional relations that relate them to fields in the supergravity multiplet $\lambda_I = \frac{1}{5!}\varepsilon_{IJ_1 \dots J_5} \chi^{J_1 \dots J_5}$ and $\lambda^{IJK} = \frac{1}{3!}\varepsilon^{IJKLMN} \chi_{LMN}$. In practice, in $\mathcal{N} = 6$, it is easier to work with λ_I and χ_{IJK} , which fit in the general pattern.

Combining this knowledge with the fermion shifts of the $\mathcal{N} = 2$ theories written above,³⁶ it is not difficult to guess the form of the generic fermion shifts:

$$\delta_\epsilon \psi_{\mu I} \sim \dots + \mathcal{V}_{IK}^M \vartheta_M^A P_A^{\mathbf{i}} (M_{\mathbf{i}})^K{}_J \gamma_\mu \epsilon^J, \quad (3.44)$$

$$\delta_\epsilon \chi_{IJK} \sim \dots + \mathcal{V}_{[IJ]}^M \vartheta_M^A P_A^{\mathbf{i}} (M_{\mathbf{i}})^L{}_{[K]} \epsilon_{L]} \quad (3.45)$$

$$\delta_\epsilon \lambda_{iI} \sim \dots + \mathcal{V}_i^M \vartheta_M^A P_A^{\mathbf{i}} (M_{\mathbf{i}})^J{}_I \epsilon_J, \quad (3.46)$$

where we have boldfaced the H indices to distinguish them from those labeling the vector supermultiplets. For the $\mathcal{N} = 3, 5$ cases there are additional fermion fields which are independent of $\psi_{\mu I}, \chi_{IJK}, \lambda_{iI}$ and whose fermion shifts are more difficult to guess. We have found the following possibilities:³⁷

1. For the $SU(3)$ singlets $\lambda_i = \frac{1}{3!}\varepsilon_{IJK} \lambda_i^{IJK}$ of $\mathcal{N} = 3$

$$\delta_\epsilon \lambda_i \sim \dots + \varepsilon_{IJK} \mathcal{V}_i^M \vartheta_M^A P_A^{\mathbf{i}} (M_{\mathbf{i}})^I{}_L \delta^{LJ} \epsilon^K. \quad (3.47)$$

2. For the $SU(5)$ singlet $\chi = \frac{1}{5!}\varepsilon_{I_1 \dots I_5} \chi^{I_1 \dots I_5}$ of $\mathcal{N} = 5$

$$\delta_\epsilon \chi \sim \dots + \varepsilon^{I_1 I_2 I_3 I_4 I_5} \mathcal{V}_{I_1 I_2}^M \vartheta_M^A P_A^{\mathbf{i}} \delta_{I_3 J} (M_{\mathbf{i}})^J{}_{I_4} \epsilon_{I_5}. \quad (3.48)$$

In many gauged supergravities (see, for instance the $\mathcal{N} = 3, d = 4$ theories [58, 59] or the $\mathcal{N} = 2, d = 8$ theories [60, 61]), the fermion shifts are given in terms of the “dressed structure constants” of the gauge group. In the $SO(3)$ -gauged $\mathcal{N} = 2, d = 8$ theory of Ref. [60], and in the conventions used there, these are defined by

$$f_{ij}{}^k \equiv L_i^m L_j^n L_p^k f_{mn}{}^p, \quad \text{where} \quad f_{mn}{}^p = \epsilon_{mnp}, \quad (3.49)$$

³⁵ χ^{IJKLM} is only relevant as an independent field for $\mathcal{N} = 5$, because it is also related to another field for $\mathcal{N} = 6$, as we are going to see.

³⁶Evidently, the fermion shifts in the hyperini will not be generalized, as there are no hypermultiplets in $\mathcal{N} \neq 2, d = 4$ theories.

³⁷We thank Mario Trigiante for enlightening conversations on this point.

and where L_i^m is the $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ coset representative ($m, n, p = 1, 2, 3$ are indices in the fundamental (vector) representation of $\text{SL}(3, \mathbb{R})$ and $i, j, k = 1, 2, 3$ are indices in the fundamental representation of $\text{SO}(3)$) and $L_m^i = (L^{-1})_m^i$ is its inverse.³⁸

The dressed structure constants can be rewritten in terms of the three momentum maps P_m^n and the three Killing vectors k_m^a (where a labels the generators of the coset) using the definition of the adjoint action of the group on the algebra, adapted to these conventions:

$$f_{ij}^k = L_i^m L_j^n (T_m)_n^p (L^{-1})_p^k = L_i^m \Gamma_{\text{Adj}}(L^{-1})_m^a (T_a)_j^k = L_i^m \left[P_m^l (T_l)_j^k - k_m^a (T_a)_j^k \right], \quad (3.50)$$

where we have used the (transposed of the) definition of the adjoint action of the group on the algebra Eq. (1.9).

Similar identities can be used in other supergravities and we hope the use of the momentum map can be of help in writing all gauged supergravity theories in a homogenous language.

3.2 Examples

3.2.1 $\mathcal{N} = 4, d = 4$ supergravity

The bosonic fields of pure $\mathcal{N} = 4, d = 4$ supergravity are the metric, the axidilaton $\tau = \chi + ie^{-2\varphi}$,³⁹ and six vector fields A_μ^Λ $\Lambda = 1, \dots, 6$. The bosonic action is

$$S = \int d^4x \sqrt{|g|} \left\{ R + \frac{\partial_\mu \tau \partial^\mu \tau^*}{2(\Im \tau)^2} - \frac{1}{4} e^{-2\varphi} F_{\mu\nu}^\Lambda F^{\Lambda\mu\nu} + \frac{1}{4} \chi F_{\mu\nu}^\Lambda \star F^{\Lambda\mu\nu} \right\}, \quad (3.51)$$

and, comparing with the generic action Eq. (3.1) we find that the period matrix is given by

$$\mathcal{N}_{\Lambda\Sigma} = -\frac{1}{8} \tau \delta_{\Lambda\Sigma}. \quad (3.52)$$

The action of this theory is invariant under $\text{SO}(6)$ rotations of the vector fields, whose symplectic infinitesimal generators, labeled by a are

$$(\mathcal{T}_a^M{}_N) = \begin{pmatrix} T_a^\Lambda{}_\Sigma & 0 \\ 0 & T_{a\Lambda}{}^\Sigma \end{pmatrix}, \quad \text{where } T_a^\Lambda{}_\Sigma = -T_{a\Sigma}{}^\Lambda. \quad (3.53)$$

The equations of motion are also invariant under the $\text{SL}(2, \mathbb{R})$ group of simultaneous electric-magnetic rotation of all the electric field strengths F^Λ into the dual magnetic

³⁸In our notation $L_i^m = u^m{}_i$, the transposed.

³⁹Observe that here $\Im \tau = e^{-2\varphi}$ instead of $e^{-\varphi}$ as in Section 1.3.1.

ones G_Λ defined in Eq. (3.3). The symplectic generators associated to these transformations are the tensor products of those in Eq. (1.73) by the identity in 6 dimensions. More explicitly

$$\begin{aligned} (\mathcal{T}_1^M{}_N) &= \frac{1}{2} \begin{pmatrix} \delta^\Lambda{}_\Sigma & 0 \\ 0 & -\delta_\Lambda{}^\Sigma \end{pmatrix}, & (\mathcal{T}_2^M{}_N) &= \frac{1}{2} \begin{pmatrix} 0 & \delta^\Lambda{}_\Sigma \\ \delta_{\Lambda\Sigma} & 0 \end{pmatrix}, \\ (\mathcal{T}_3^M{}_N) &= \frac{1}{2} \begin{pmatrix} 0 & \delta^\Lambda{}_\Sigma \\ -\delta_{\Lambda\Sigma} & 0 \end{pmatrix}. \end{aligned} \quad (3.54)$$

We will denote the generators of $SL(2, \mathbb{R})$ with the label α to distinguish them from those of the $SO(6)$ group. Observe that there are no scalars associated to $SO(6)$. The only scalar, the axidilaton, is invariant under $SO(6)$. Since this group is a symmetry of the action, the NGZ current coincides with the Noether current, has no scalar contribution and is given by

$$j_a = -2\mathcal{T}_a^M{}_N \star (\mathcal{F}^N \wedge \mathcal{A}_M). \quad (3.55)$$

Using Maxwell equations and Bianchi identities of the vector fields we find that

$$d \star j_a = -2\mathcal{T}_a^M{}_N \mathcal{F}^N \wedge \mathcal{F}_M, \quad (3.56)$$

which vanishes identically due to the antisymmetry of the $SO(6)$ generators. In this case, evidently, there is nothing to be dualized and there are no 2-forms B_a .

The NGZ currents of the $SL(2, \mathbb{R})$ electric-magnetic duality group are non-trivial, though:

$$j_\alpha = j_\alpha^{(\sigma)} - 2\mathcal{T}_\alpha^M{}_N \star (\mathcal{F}^N \wedge \mathcal{A}_M), \quad \alpha = 1, 2, 3. \quad (3.57)$$

It is necessary to use the equations of motion of the scalars (and not just the Maxwell equations and Bianchi identities) to show that they are conserved on-shell. They are dualized into 3 2-forms B_α according to the general prescription. We will not give the details here.

3.2.2 $\mathcal{N} = 8, d = 4$ supergravity

The above general scheme can be applied to $\mathcal{N} = 8, d = 4$ supergravity even if we use a complex basis for the vector fields. Thus, we have

$$H_E{}_F = dB_E{}_F - 4 \left[\mathcal{F}^{EA} \mathcal{A}_{FA} + \mathcal{A}^{EA} \mathcal{F}_{FA} - \frac{1}{8} \delta^E{}_F \left(\mathcal{F}^{AB} \mathcal{A}_{AB} + \mathcal{A}^{AB} \mathcal{F}_{AB} \right) \right], \quad (3.58)$$

$$H_{EFGH} = dB_{EFGH} - 2 \left(\mathcal{F}_{[EF} \mathcal{A}_{GH]} + \frac{1}{4!} \varepsilon_{EFGHABCD} \mathcal{F}^{AB} \mathcal{A}^{CD} \right). \quad (3.59)$$

4 The higher-dimensional, higher-rank NGZ 1-forms and dualization

In $d > 4$ dimensions supergravity theories may contain dynamical fields which are differential forms of rank $p > 1$. The global symmetries of the theory can act on these field as rotations or, when the rank and dimension allow it ($d = 2(p + 1)$), as electric-magnetic transformations. The latter are not symmetries of the action but, nevertheless, as in the 4-dimensional case, a generalized Noether-Gaillard-Zumino (NGZ) current 1-form j_A which is conserved on shell can be defined for each and all the generators of the full duality group.

The equation that expresses this conservation can be written in a universal form: let F^I, H_m, G^a, \dots be, respectively, the 2-, 3-, 4-, ... form field strengths of the n_1, n_2, n_3, \dots fundamental 1-, 2-, 3-, ... fields of the theory and let $\tilde{F}_I, \tilde{H}^m, \tilde{G}_a, \dots$ their dual $(d - 2)$ -, $(d - 3)$ -, $(d - 4)$ -, ... form field strengths. As the indices chosen show, if the fundamental field strengths F^I, H_m, G^a, \dots transform linearly under the duality group as $\delta_A F^I = T_A^I{}_J F^J$, $\delta_A H_m = -T_A^m{}_n H_n$, $\delta_A G^a = T_A^a{}_b G^b$, ..., the dual field strengths must transform in the conjugate representations, that is $\delta_A \tilde{F}_I = -T_A^I{}_J \tilde{F}_J$, $\delta_A \tilde{H}^m = T_A^m{}_n \tilde{H}^n$, $\delta_A \tilde{G}_a = -T_A^a{}_b \tilde{G}_b$, ... The only exception to these transformation rules are the electric-magnetic transformations. In $d = 4$, for instance, they relate F^I to \tilde{F}_I and the pair $(\mathcal{F}^M) \equiv \begin{pmatrix} F^I \\ \tilde{F}_I \end{pmatrix}$ transforms as a $\text{Sp}(2n_1, \mathbb{R})$ vector according to $\delta_A \mathcal{F}^M = T_A^M{}_N \mathcal{F}^N$ with $T_A^M{}_N \in \mathfrak{sp}(2n_1, \mathbb{R})$. In $d = 6$, electric-magnetic duality transformations relate H_m to \tilde{H}^m and the pair $(\mathcal{H}^M) \equiv \begin{pmatrix} H_m \\ \tilde{H}^m \end{pmatrix}$ transforms as a $\text{SO}(n_2, n_2)$ vector according to $\delta_A \mathcal{H}^M = T_A^M{}_N \mathcal{H}^N$ with $T_A^M{}_N \in \mathfrak{so}(n_2, n_2)$ etc.

It is not difficult to see through the 5- and 8-dimensional examples we are going to present next that the equation satisfied by the Noether current 1-forms is always, up to conventional coefficients, of the form

$$-k_A^x \frac{\delta S}{\delta \phi^x} = d \star j_A + T_A^I{}_J F^J \wedge \tilde{F}_I + T_A^m{}_n \tilde{H}^n \wedge H_m \cdots = 0, \quad (4.1)$$

and in the exceptional cases mentioned above, one should replace $T_A^I{}_J F^J \wedge \tilde{F}_I$ by $\frac{1}{2} T_A^M{}_N \mathcal{F}_M \wedge \mathcal{F}^N$, $T_A^m{}_n \tilde{H}^n \wedge H_m$ by $\frac{1}{2} T_A^M{}_N \mathcal{H}_M \wedge \mathcal{H}^N$ etc.

On-shell, the above equation would take the form

$$d \star j_A^{\text{NGZ}} = 0, \quad (4.2)$$

but it is not possible to give a general form of this current because, in each theory, the field strengths contain different Chern-Simons terms, all of them duality-invariant. In the 5-dimensional example that follows, we have found the explicit form, but in the 8-dimensional one, we have not.

The dualization of the NGZ current 1-forms into $(d - 2)$ -form potentials proceeds as in the 4-dimensional case.

4.1 Examples

4.1.1 $\mathcal{N} = 1, d = 5$ supergravities

The bosonic action of any 5-dimensional ungauged supergravity-like theory with scalars ϕ^x and Abelian vector fields A^I (in particular, $\mathcal{N} = 1, d = 5$ supergravities with vector supermultiplets) can be written in the form [46]

$$S = \int \left\{ \star R + \frac{1}{2} \mathcal{G}_{xy} d\phi^x \wedge \star d\phi^y - \frac{1}{2} a_{IJ} F^I \wedge \star F^J + \frac{1}{3} C_{IJK} F^I \wedge F^J \wedge A^K \right\}, \quad (4.3)$$

where $\mathcal{G}_{xy}(\phi)$ is the σ -model metric, $a_{IJ}(\phi)$ is the kinetic matrix of the vector fields and C_{IJK} is a constant, symmetric tensor. In supergravity theories these three couplings are related in a very precise way, but we will not need to use this structure for our purposes.

The equations of motion of the vector fields are

$$d(a_{IJ} \star F^J - C_{IJK} F^J \wedge A^K) = 0, \quad (4.4)$$

and can be solved locally by

$$a_{IJ} \star F^J - C_{IJK} F^J \wedge A^K \equiv d\tilde{A}_I, \quad (4.5)$$

where the \tilde{A}_I are the magnetic 2-forms dual to the vector fields. Their gauge-invariant field strengths are

$$\tilde{F}_I = d\tilde{A}_I + C_{IJK} F^J \wedge A^K, \Rightarrow d\tilde{F}_I = C_{IJK} F^J \wedge F^K, \quad (4.6)$$

and are related to the vector field strengths by

$$\tilde{F}_I = a_{IJ} \star F^J. \quad (4.7)$$

The equations of motion of the scalars are

$$-\frac{\delta S}{\delta \phi^z} = \mathcal{G}_{zw} [d \star d\phi^w + \Gamma_{xy}{}^w d\phi^x \wedge \star d\phi^y] + \frac{1}{2} \partial_z a_{IJ} F^I \wedge \star F^J. \quad (4.8)$$

If the action is invariant under the global transformations generated by

$$\begin{aligned} \delta_A \phi^x &= k_A^x(\phi), \\ \delta_A A^I &= T_A^I{}_J A^J, \end{aligned} \quad (4.9)$$

which implies that the functions $k_A^x(\phi)$ are Killing vectors of the σ -model metric \mathcal{G}_{xy} , the kinetic matrix satisfies

$$k_A^x \partial_x a_{IJ} = -2 T_A^K{}_{(I} a_{J)K}, \quad (4.10)$$

and the symmetric tensor satisfies

$$T_A{}^L{}_{(I}C_{JK)L} = 0, \quad (4.11)$$

we find that

$$-k_A{}^z \frac{\delta S}{\delta \phi^z} = d \star j_A^{(\sigma)} - T_A{}^K{}_I a_{JK} F^I \wedge \star F^J. \quad (4.12)$$

In order to dualize the Noether currents, we first have to replace the Hodge dual of the vector field strengths by the \tilde{F}_I :

$$-k_A{}^z \frac{\delta S}{\delta \phi^z} = d \star j_A^{(\sigma)} - T_A{}^K{}_I F^I \wedge \tilde{F}_K, \quad (4.13)$$

and, then, using the invariance of the C_{IJK} tensor Eq. (4.11) we get

$$-k_A{}^z \frac{\delta S}{\delta \phi^z} = d \left[\star j_A^{(\sigma)} - \frac{1}{3} T_A{}^K{}_I (A^I \wedge \tilde{F}_K + 2F^I \wedge \tilde{A}_K) \right] = 0. \quad (4.14)$$

As usual, we solve locally this equation by introducing 3-form potentials D_A

$$\star j_A^{(\sigma)} - \frac{1}{3} T_A{}^K{}_I (A^I \wedge \tilde{F}_K + 2F^I \wedge \tilde{A}_K) \equiv dD_A, \quad (4.15)$$

with gauge-invariant field strengths and duality relation

$$K_A = dD_A - \frac{1}{3} T_A{}^K{}_I (A^I \wedge \tilde{F}_K + 2F^I \wedge \tilde{A}_K), \quad \star j_A^{(\sigma)} = K_A. \quad (4.16)$$

4.1.2 $\mathcal{N} = 2, d = 8$ supergravity

This example is based on the results found in Ref. [62]. The possible electric fields in an 8-dimensional theory are scalars ϕ^x , 1-forms A^I , 2-forms B_m , and 3-forms C^a . The most general Abelian, massless, ungauged supergravity-like theory in 8 dimensions with this field content can be written in the form

$$\begin{aligned} S = & \int \left\{ -\star R + \frac{1}{2} \mathcal{G}_{xy} d\phi^x \wedge \star d\phi^y + \frac{1}{2} \mathcal{M}_{IJ} F^I \wedge \star F^J + \frac{1}{2} \mathcal{M}^{mn} H_m \wedge \star H_n \right. \\ & - \frac{1}{2} \Im \mathcal{N}_{ab} G^a \wedge \star G^b - \frac{1}{2} \Re \mathcal{N}_{ab} G^a \wedge G^b \\ & - dC^a \wedge \Delta G_a - \frac{1}{2} \Delta G^a \wedge \Delta G_a - \frac{1}{6} d^{mnp} B_m \wedge dB_n \wedge dB_p + \frac{1}{2} d^{mnp} B_m \wedge H_n \wedge H_p \\ & \left. + \frac{1}{24} d^i{}_I{}^m d_{ij}{}^n A^I \wedge A^J \wedge \Delta H_m \wedge dB_n \right\}. \end{aligned} \quad (4.17)$$

where $\mathcal{G}_{xy}, \mathcal{M}_{IJ}, \mathcal{M}^{mn}, \mathcal{N}_{ab}$ are scalar-dependent kinetic matrices (\mathcal{N}_{ab} complex and the rest real), the field strengths are defined by

$$F^I = dA^I. \quad (4.18)$$

$$H_m = dB_m - d_{mIJ}F^I \wedge A^J, \quad (4.19)$$

$$G^a = dC^a + d^a_{I^m}F^I \wedge B_m - \frac{1}{3}d^a_{I^m}d_{mJK}A^I \wedge F^J \wedge A^K, \quad (4.20)$$

$d_{mIJ}, d^a_{I^m}$ being constant deformation parameters and ΔG^a etc. denote all the terms in the corresponding field strength but dC^a etc.

The 3-forms can be dualized in 3-forms C_a with field strengths and duality relations

$$G_a \equiv dC_a + d_{aI^m}F^I \wedge B_m - \frac{1}{3}d_{aI^m}d_{mJK}A^I \wedge F^J \wedge A^K, \quad (4.21)$$

$$G_a = -\Im \mathcal{N}_{ab}G^a \wedge \star G^b - \Re \mathcal{N}_{ab}G^a \wedge G^b \equiv R_a,$$

where the d_{aI^m} are constant independent parameters. The electric and magnetic 3-forms and the deformation parameters can be collected in symplectic vectors:

$$(C^i) \equiv \begin{pmatrix} C^a \\ C_a \end{pmatrix}, \quad (d^i_{I^m}) \equiv \begin{pmatrix} d^a_{I^m} \\ d_{aI^m} \end{pmatrix}, \quad (4.22)$$

with the symplectic indices i, j to be raised and lowered with the symplectic metric $(\Omega_{ij}) = (\Omega^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The 2-forms B_n can be dualized into 4-forms \tilde{B}^m with field strength and duality relation

$$\begin{aligned} \tilde{H}^m &\equiv d\tilde{B}^m + d^i_{I^m}C_i \wedge F^I + d^{mnp}B_n \wedge (H_p + \Delta H_p) + \frac{1}{12}d^i_{I^m}d_{ij}^n A^I \wedge A^J \wedge \Delta H_n, \\ \tilde{H}^m &= \mathcal{M}^{mn} \star H_n, \end{aligned} \quad (4.23)$$

where the new deformation $d^{mnp} = d^{[mnp]}$ must be related to the other deformations by

$$d^i_{(I|}{}^m d_{i|J)}{}^n = -2d^{mnp}d_{pIJ}. \quad (4.24)$$

Finally, the 1-forms A^I can be dualized into 6-forms \tilde{A}_I with field strength and duality relation

$$\begin{aligned} \tilde{F}_I &\equiv d\tilde{A}_I + \dots, \\ \tilde{F}_I &= \mathcal{M}_{IJ} \star F^J, \end{aligned} \quad (4.25)$$

where the dots stand for a very long expression that can be found in Ref. [62].

As in the previous example, let us assume that the equations of motion are invariant under the global transformations generated by⁴⁰

$$\begin{aligned}\delta_A \phi^x &= k_A^x(\phi), & \delta_A A^I &= T_A^I{}_J A^J, \\ \delta_A B_m &= -T_A^n{}_m B_n, & \delta_A C^i &= T_A^i{}_j C^j,\end{aligned}\tag{4.26}$$

where the matrices $T_A^I{}_J$, $T_A^m{}_n$ and the matrices

$$(T_A^i{}_j) \equiv \begin{pmatrix} T_A^{ab} & T_A^{ab} \\ T_{Aab} & T_{Aa}{}^b \end{pmatrix},\tag{4.27}$$

which must be generators of the symplectic group,

$$T_A^i{}_{[j} \Omega_{k]i} = 0,\tag{4.28}$$

are different representations of the same Lie algebra as the one generated by the vectors $k_A^x(\phi)$:

$$[T_A, T_B] = f_{AB}{}^C T_C, \quad [k_A, k_B] = -f_{AB}{}^C k_C.\tag{4.29}$$

As in the previous case, this implies that the functions $k_A^x(\phi)$ are Killing vectors of the σ -model metric \mathcal{G}_{xy} , the kinetic matrices satisfy⁴¹

$$\begin{aligned}k_A^x \partial_x \mathcal{M}_{IJ} &= -2T_A^K ({}_I \mathcal{M}_{J)K}, \\ k_A^x \partial_x \mathcal{M}^{mn} &= 2T_A^{(m}{}_p \mathcal{M}^{n)p},\end{aligned}\tag{4.30}$$

$$k_A^x \partial_x \mathcal{N}_{ab} = -T_{Aab} - \mathcal{N}_{ac} T_A^c{}_b + T_{Aa}{}^c \mathcal{N}_{cb} + \mathcal{N}_{ac} T_A^{cd} \mathcal{N}_{db},$$

and the deformation tensors d_{mIJ} , $d^i{}_I{}^m$, d^{mnp} are invariant under the δ_A transformations:

$$\begin{aligned}\delta_A d_{mIJ} &= -T_A^n{}_m d_{nIJ} - 2T_A^K ({}_I d_{n|J)K} = 0, \\ \delta_A d^i{}_I{}^m &= T_A^i{}_j d^j{}_I{}^m - T_A^J{}_I d^i{}_J{}^m + T_A^m{}_n d^i{}_I{}^n = 0, \\ \delta_A d^{mnp} &= 3T_A^{[m}{}_q d^{np]q} = 0.\end{aligned}\tag{4.31}$$

Since, in general, these symmetries are not symmetries of the action, we proceed as in the 4-dimensional case, contracting the equations of motion of the scalars, given by

⁴⁰Observe that the transformations involving the 3-forms include electric-magnetic rotations. 3-forms in 8 dimensions transform as the 1-forms in 4-dimensions with groups which must be embedded in the symplectic group.

⁴¹The transformation rule of the period matrix is unusual because our definition of the dual 4-form field strength differs by a sign from the usual one.

$$-\frac{\delta S}{\delta \phi^x} = \mathcal{G}_{xy} [d \star d\phi^y + \Gamma_{zw}^y d\phi^z \wedge \star d\phi^w] \quad (4.32)$$

$$-\frac{1}{2}\partial_x \mathcal{M}_{IJ} F^I \wedge \star F^J - \frac{1}{2}\partial_x \mathcal{M}^{mn} H_m \wedge \star H_n - G^a \partial_x R_a,$$

with the Killing vectors of the σ -model metric $\mathcal{G}_{xy}(\phi)$, $k_A^x(\phi)$. Using the Killing equation, we get

$$-k_A^x \frac{\delta S}{\delta \phi^x} = d \star j_A^{(\sigma)} - \frac{1}{2} k_A^x \partial_x \mathcal{M}_{IJ} F^I \wedge \star F^J - \frac{1}{2} k_A^x \partial_x \mathcal{M}^{mn} H_m \wedge \star H_n - G^a k_A^x \partial_x R_a. \quad (4.33)$$

Using now Eqs. (4.30) and the duality relations for the field strengths, we arrive to

$$-k_A^x \frac{\delta S}{\delta \phi^x} = d \star j_A^{(\sigma)} + T_A^I F^I \wedge \tilde{F}_J + T_A^m \tilde{H}^n \wedge H_m + \frac{1}{2} T_A^i G^j \wedge G_j = 0, \quad (4.34)$$

on shell. It is not difficult to see that the exterior derivative of the expression in the l.h.s. of the equation vanishes due to the Bianchi identities satisfied by the field strengths and due to the invariance of the deformation tensors expressed in the relations Eqs. (4.31). This means that it should be possible to rewrite this equation as the conservation of a higher-dimensional NGZ current, that is

$$d \star j_A^{NGZ} = 0, \quad j_A^{NGZ} \equiv j_A^{(\sigma)} + \Delta j_A, \quad (4.35)$$

where Δj_A has a very long a complicated form.

A local solution of this conservation equation is provided by $\star[j_A^{(\sigma)} + \Delta j_A] = dD_A$ where D_A is a 6-form potential D_A . Then, reasoning as in the 4-dimensional case, the gauge-invariant 7-form field strength K_A and its duality relation will be given by

$$K_A \equiv dD_A + \star \Delta j_A, \quad K_A = \star j_A^{(\sigma)}, \quad (4.36)$$

and its Bianchi identity will adopt the universal form

$$dK_A = T_A^I F^I \wedge \tilde{F}_I + T_A^m \tilde{H}^n \wedge H_m + \frac{1}{2} T_A^i G^j \wedge G_j. \quad (4.37)$$

4.1.3 $\mathcal{N} = 2B, d = 10$ supergravity

Our last example concerns the dualization of the scalars of $\mathcal{N} = 2B, d = 10$ supergravity [34, 35, 36], the effective field theory of the type IIB superstring. They are the dilaton φ and the RR 0-form χ and, combined in the axidilaton $\tau = \chi + ie^{-\varphi}$ they parametrize the $SL(2, \mathbb{R})/SO(2)$ described in Section 1.3.1. They are dualized into 3 8-form potentials satisfying a constraint [13, 9] according to the general rules and the

field strengths, whose form depends very strongly on conventions, satisfy a Bianchi identity of the universal form proposed above.

Here we want to focus on the supersymmetry transformation rules of the 8-forms, constructed in Refs. [9, 38] in the $SU(1, 1)/U(1)$ formulation used in [35] and studied in Section 1.3.2. We want to compare them with the general form proposed in Section 3.1. They are given by

$$\begin{aligned} \delta_\epsilon A^{\alpha\beta}_{\mu_1\cdots\mu_8} &= 8V^{(\alpha} V^{\beta)} \bar{\epsilon} \gamma_{[\mu_1\cdots\mu_7} \psi_{\mu_8]} + \text{c.c.} \\ &\quad - iV^\alpha V^\beta \bar{\epsilon} \gamma_{\mu_1\cdots\mu_8} \lambda_C + \text{c.c.} \\ &\quad + \cdots \end{aligned} \tag{4.38}$$

where we have omitted terms proportional to other p -form fields, which are related to the Chern-Simons terms in the 9-form field strengths. Comparing now with Eqs. (1.106) and (1.107) we see that the terms constraining the gravitini are multiplied by the momentum map while the terms containing the dilatini are proportional to the Killing vectors, as expected according to our general arguments.

5 Conclusions

In this paper we have reviewed the general problem of dualizing the scalars of a d -dimensional theory into $(d - 2)$ -form potentials preserving the dualities of the theory in a manifest form and taking into account their possible couplings to the potentials of the theory. We have not considered the dualization in presence of a scalar potential, since doing this properly, requires the full tensor hierarchy machinery, which lies outside of the scope of this paper.⁴²

In general, the dualization procedure has to be necessarily incomplete: the non-linearly interacting scalars cannot be replaced completely by the $(d - 2)$ -form potentials, as often happens in supergravity theories with most potentials. Nevertheless, one may hope to find a PST-like formulation for them. For the particular case of scalars parametrizing the coset $SU(1, 1)/U(1)$ the PST-type action was constructed in Ref. [9] as a part of type IIB supergravity action. Here we have presented the generalization of the action of Ref. [9] for the generic symmetric space G/H ; the properties of this action and its applications will be considered elsewhere.

Since we need to dualize conserved charges and some of the symmetries one has to consider in supergravity theories leave invariant the equations of motion but not the action, it is necessary to consider the Noether-Gaillard-Zumino current, whose generalization to theories in higher dimensions and with higher-rank potentials we have studied.

⁴²The general $d = 8$ case studied in Ref. [62] provides a quite complete example of how to proceed in that case.

During this study we have found it necessary to extend the concept of momentum map to all symmetric spaces. The holomorphic and triholomorphic momentum maps defined in Kähler and quaternionic-Kähler spaces play a very important rôle in $\mathcal{N} = 1, 2, d = 4$ supergravities: they occur in fermion shifts (and, therefore, in the scalar potentials, where they often appear disguised as “dressed structure constants” or “T-tensors”), in the supersymmetry transformations of the 2-forms dual to the scalars (and, therefore, in the tensions of the strings that couple to them) and in the covariant derivatives of the fermions in gauged supergravities. We have shown through examples that the generalized momentum map satisfies similar equations and plays exactly the same rôle in $\mathcal{N} > 2$ and $d > 4$ supergravities and we have explored the general form of the supersymmetry transformation rules of the $(d - 2)$ -forms dual to the scalars and the fermion shifts.

In $\mathcal{N} = 1$ supersymmetric mechanics one can consider general manifolds with no special holonomy properties. When they admit isometries and we gauge them, the covariant derivatives of the fermions contain an object that plays the same rôle as the momentum map. We have shown that it satisfies analogous equations and that, when the manifold has special holonomy (Kähler, quaternionic-Kähler or symmetric space), this object is the (generalized) momentum map. We have, therefore, proposed a more fundamental definition for the momentum map that encompasses all the previous ones.

Supergravity theories have very different forms for different values of \mathcal{N}, d , mostly because of historical reasons: some of them have been constructed by dimensional reduction, some others in superspace or using other approaches. This complicates unnecessarily working with them and establishing relations between them via compactifications, truncations, gaugings etc. As a theories of dynamical supergeometry, it should be possible to describe them in a more \mathcal{N} - and d -independent form. A big step in this direction was taken in Ref. [57], specially for 4-dimensional theories, which were described in an almost \mathcal{N} -independent fashion, but neither the gaugings nor the higher-rank form fields were considered there. We hope the extension of the concept of momentum map proposed here and its systematic use (specially in the construction of fermion shifts and scalars potentials) will be useful to rewrite all gauged supergravities in a more homogeneous form.

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A Kähler–Hodge manifolds in $\mathcal{N} = 1, 2, d = 4$ supergravity

In this Appendix we want to review briefly the definition of the holomorphic momentum map and other structures which have their parallel in the main text in the context of Kähler–Hodge (KH) manifolds, which are not necessarily symmetric or even homogenous spaces. We adopt the notation and conventions of Refs. [26, 18].

A Kähler manifold is a complex, Hermitian manifold whose fundamental 2-form \mathcal{J}

$$\mathcal{J} \equiv \mathcal{J}_{ij^*} dZ^i \wedge dZ^{*j^*} = 2i\mathcal{G}_{ij^*} dZ^i \wedge dZ^{*j^*}, \quad (\text{A.1})$$

is closed

$$d\mathcal{J} = 0. \quad (\text{A.2})$$

This equation implies the vanishing of the torsion, the identification of the Hermitian connection with the Levi-Civita connection and the local existence of a real function, the Kähler potential $\mathcal{K}(Z, Z^*)$, such that

$$\frac{1}{2i}\mathcal{J}_{ij^*} = \mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}. \quad (\text{A.3})$$

\mathcal{K} is defined up to Kähler transformations, which have the form

$$\mathcal{K}' = \mathcal{K} + \lambda(Z) + \lambda^*(Z^*), \quad (\text{A.4})$$

where $\lambda(Z)$ is an arbitrary holomorphic function of the complex coordinates Z^i .

In $\mathcal{N} = 1, 2, d = 4$ supergravity there are complex scalar field parametrizing Kähler manifolds and the Kähler metric \mathcal{G}_{ij^*} plays the role of the σ -model metric.

The Kähler (connection) 1-form is defined by

$$\mathcal{Q} \equiv \frac{1}{2i}(\partial_i \mathcal{K} dZ^i - \text{c.c.}), \quad (\text{A.5})$$

transforms under Kähler transformations as a $U(1)$ connection

$$\mathcal{Q}' = \mathcal{Q} + \frac{1}{2i}(\partial\lambda - \partial^*\lambda^*), \quad (\text{A.6})$$

and the Kähler 2-form can be seen as its Kähler-invariant curvature

$$\mathcal{J} \equiv 2d\mathcal{Q}. \quad (\text{A.7})$$

A Kähler–Hodge manifold is a Kähler manifold \mathcal{M} on which a complex line bundle $L^1 \rightarrow \mathcal{M}$ has been defined such that its first Chern class (given by the Ricci 2-form \mathfrak{R} of the fiber’s Hermitian metric) is equal to the Kähler 2-form \mathcal{J} .

As we are going to show, in the KH manifolds of $\mathcal{N} = 1, 2$, $d = 4$ supergravity, the Kähler 1-form connection \mathcal{Q} and its curvature \mathcal{J} play the same as the H= U(1) connection ϑ and its curvature $R(\vartheta)$ defined in Eqs. (1.13) and (1.18):

$$\mathcal{Q} \rightarrow -\frac{1}{2}\vartheta, \quad \mathcal{J} \rightarrow -R(\vartheta), \quad (\text{A.8})$$

even though there is no coset structure. The requirement that the Kähler manifold is actually Kähler–Hodge is crucial.

The fermionic fields of $\mathcal{N} = 1, 2$ supergravity are sections of the associated U(1) bundle, which means that, under Kähler transformations Eq. (A.4), they transform as

$$\psi' = e^{-\frac{q}{2}(\lambda - \lambda^*)} \psi, \quad (\text{A.9})$$

if their weight is the real number q . The Kähler-covariant derivative on fields of Kähler weight q is given by

$$\mathcal{D}\psi = d\psi + iq\mathcal{Q}\psi, \quad (\text{A.10})$$

where here \mathcal{Q} is the spacetime pullback of the Kähler 1-form. This definition should be compared with that of the H-covariant derivative Eq. (1.30).

Let us now assume that the theory we are considering has some global symmetry transformation group acting on the scalars. These transformations must necessarily be holomorphic isometries of the Kähler metric generated by Killing vectors $K_A \equiv k_A^i(Z)\partial_i + k_A^{*i^*}(Z^*)\partial_{i^*}$ but they must also preserve the entire KH structure.

First of all, this implies that the transformations generated by the Killing vectors will leave the Kähler potential invariant up to Kähler transformations:

$$\mathcal{L}_{k_A}\mathcal{K} \equiv k_A^i\partial_i\mathcal{K} + k_A^{*i^*}\partial_{i^*}\mathcal{K} = \lambda_A(Z) + \lambda_A^*(Z^*), \quad (\text{A.11})$$

for certain holomorphic functions $\lambda_A(Z)$. This, in its turn, implies that all the fields which transform as in Eq. (A.9) will transform as

$$\psi' = e^{-\frac{q}{2}(\lambda_A - \lambda_A^*)} \psi, \quad (\text{A.12})$$

under the transformation generated by K_A . This is similar to the H compensating transformations described in Section 1.1.1 and it is clear that the imaginary part of the holomorphic functions λ_A plays the same role as the H-compensator defined in Eq. (1.27)

$$\frac{1}{2i}(\lambda_A - \lambda_A^*) \rightarrow -2W_A. \quad (\text{A.13})$$

Taking another Lie derivative in Eq. (A.11) we find the following equivariance property

$$\mathcal{L}_{k_A} \lambda_B - \mathcal{L}_{k_B} \lambda_A = -f_{AB}{}^C \lambda_C, \quad (\text{A.14})$$

which is identical to that of the H-compensator Eq. (1.51).

Secondly, the Kähler 2-form \mathcal{J} must also be preserved

$$\mathcal{L}_{k_A} \mathcal{J} = i_{k_A} d\mathcal{J} + d(i_{k_A} \mathcal{J}) = 0. \quad (\text{A.15})$$

Eq. (A.2) and the above equation imply the local existence of real functions \mathcal{P}_A (the holomorphic momentum maps) such that

$$i_{k_A} \mathcal{J} = d\mathcal{P}_A. \quad (\text{A.16})$$

Comparing this equation with Eq. (1.34) and taking into account the correspondences Eq. (A.8) we find that the holomorphic momentum map plays the same role as the momentum map defined in Eq. (1.28):

$$\mathcal{P}_A \rightarrow \frac{1}{2} P_A. \quad (\text{A.17})$$

A local solution of Eq. (A.16) is

$$i\mathcal{P}_A = k_A^i \partial_i \mathcal{K} - \lambda_A = -(k_A^{*i} \partial_{i^*} \mathcal{K} - \lambda_A^*) = i_{k_A} \mathcal{Q} - \frac{1}{2i} (\lambda_A - \lambda_A^*), \quad (\text{A.18})$$

where we have taken into account Eq. (A.11). This equation should be compared with Eq. (1.27) that relates the H-connection, the H-compensator and the momentum map.

Furthermore, the holomorphic Killing vectors can be obtained from the momentum map (*Killing prepotential*)

$$\partial_i \mathcal{P}_A = i k_{Ai}^*. \quad (\text{A.19})$$

In $\mathcal{N} = 2, d = 4$ supergravity theories, the Special Kähler structure allows us to find a general expression for the holomorphic momentum map in terms of the covariantly holomorphic symplectic section \mathcal{V}^M and the symplectic generators $\mathcal{T}_A{}^M{}_N$:

$$\mathcal{P}_A = \langle \mathcal{V}^* | \mathcal{T}_A \mathcal{V} \rangle = \mathcal{T}_A{}^M{}_N \mathcal{V}_M^* \mathcal{V}^N. \quad (\text{A.20})$$

If we now gauge the group of holomorphic isometries generated by the Killing vectors k_A^i we can follow the same rules as in symmetric spaces to construct the gauge-covariant derivatives, adding to the pullback of the H-connection (Kähler connection) the product $A^A \mathcal{P}_A$ where A^A is the spacetime gauge field:

$$\mathcal{Q} \rightarrow \mathcal{Q} - g A^A \mathcal{P}_A = \mathcal{Q}_i \mathcal{D} Z^i + \mathcal{Q}_{i^*} \mathcal{D} Z^{*i^*} - g A^A \Im \lambda_A. \quad (\text{A.21})$$

The momentum map also occurs in the fermion shifts of the fermions' supersymmetry transformation rules. The details depend on the theory and its R-symmetry group and can be found, for instance, in Ref. [18].

A.1 2-form potentials from the Kähler–Hodge manifolds of $\mathcal{N} = 1, 2$, $d = 4$ supergravity

The dualization of the complex scalars of $\mathcal{N} = 1$ and $\mathcal{N} = 2, d = 4$ supergravities belonging to chiral and vector supermultiplets can be performed following the general procedure outlined in Section 3. To finish the job, though, a supersymmetry transformation rule must be provided for the dual 2-form fields, at least to lowest (zeroth) order in fermions. The supersymmetry algebra must close on shell and up to duality relations between the magnetic and electric vector fields and between the 2-forms and the NGZ currents.

This was first done in the $\mathcal{N} = 2, d = 4$ theories in Ref. [45]. After the use of the expression for the momentum maps Eq. (A.20), the supersymmetry transformation rules found there can be written in the form

$$\delta_\epsilon B_{A\mu\nu} = -i\mathcal{P}_A \bar{\epsilon}^I \gamma_{[\mu} \psi_{I\nu]} - \frac{i}{2} k_A^* \bar{\epsilon}^I \gamma_{\mu\nu} \lambda^{iI} + \text{c.c.} + 8\mathcal{T}_A^M \mathcal{A}_{M[\mu} \delta_\epsilon \mathcal{A}_{N|\nu]}. \quad (\text{A.22})$$

The commutator of two of these supersymmetry transformations gives

$$[\delta_\eta, \delta_\epsilon] = \delta_{\text{g.c.t.}}(\tilde{\zeta}) + \delta_{\text{gauge}}(\Lambda) + \delta_{\text{gauge}}(\Lambda_1). \quad (\text{A.23})$$

where $\tilde{\zeta}^\mu$ are the parameters of general coordinate transformations, Λ^M are the 0-form parameters of the gauge transformations of the gauge fields \mathcal{A}^M_μ and Λ_{1A} are the 1-form parameters of the gauge transformations of the 2-form fields $B_{A\mu\nu}$. Their explicit expressions can be found in Ref. [45].

In the actual computation of the commutator, the derivative of the momentum map, which gives the corresponding Killing vector and the scalar part of the NGZ current appears naturally. Upon dualization, that term gives the contraction of the 3-form field strength H_A with $\tilde{\zeta}^\mu$, which is a general coordinate transformation of B_A up to a gauge transformation.

The supersymmetry transformation rule for the 2-forms of $\mathcal{N} = 1, d = 4$ supergravity was given in Ref. [46] and fits into the same pattern (the differences are basically due to the different conventions)

$$\delta_\epsilon B_{A\mu\nu} = \frac{i}{2} \mathcal{P}_A \bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} + \frac{1}{4} \partial_i \mathcal{P}_A \bar{\epsilon} \gamma_{\mu\nu} \chi^i + \text{c.c.} - 2\mathcal{T}_A^M \mathcal{A}_{M[\mu} \delta_\epsilon \mathcal{A}_{\nu]}^N. \quad (\text{A.24})$$

B Quaternionic–Kähler manifolds in $\mathcal{N} = 2, d = 4$ supergravity

The structures constructed for symmetric spaces can also be generalized to quaternionic–Kähler (QK) spaces, $4m$ -dimensional Riemannian spaces whose holonomy group is $\text{SU}(2) \times \text{Sp}(2m)$.

A QK manifold is a $4m$ -dimensional Riemannian manifold that satisfies the following properties:

1. It admits a triplet of complex structures J_m^x , $x = 1, 2, 3$ satisfying the algebra of the unit imaginary quaternions

$$J_m^x J_p^y = -\delta^{xy} \delta_m^n + \varepsilon^{xyz} J_m^z. \quad (\text{B.1})$$

(Observe that this property implies the property that characterizes complex structures $(J^x)^2 = -1$, $\forall x$.)

2. The Riemannian metric g_{mn} is Hermitian with respect to the three complex structures:

$$g_{mn} = J_m^{(x)p} J_n^{(x)q} g_{pq}. \quad (\text{B.2})$$

We can define a triplet of Kähler 2-forms (hyperKähler 2-form)

$$J_{mn}^x \equiv J_m^x J_n^p g_{np}. \quad (\text{B.3})$$

3. There is a $SU(2)$ bundle over the QK space with connection 1-form $A_m^x d\phi^m$ and it is required that the hyperKähler 2-form is covariantly constant with respect to it:

$$D_m J_{np}^x \equiv \nabla_m(\omega) J_{np}^x + \varepsilon^{xyz} A_m^y J_{np}^z = 0, \quad (\text{B.4})$$

where $\nabla_m(\omega)$ stands for the covariant derivative with the Levi-Civita connection ω .

4. The $SU(2)$ curvature, defined by

$$F^x \equiv dA^x + \frac{1}{2} \varepsilon^{xyz} A^y \wedge A^z, \quad (\text{B.5})$$

is proportional to the hyperKähler structure

$$F^x = \varkappa J^x. \quad (\text{B.6})$$

In $\mathcal{N} = 2, d = 4$ $\varkappa = -1$. (If $\varkappa = 0$ the manifold is just a hyperKähler manifold).

This last property of QK manifolds combined with the relation between the $SU(2)$ component of the curvature 2-form of the Levi-Civita connection (obtained through the projection with the hyperKähler structure) and the hyperKähler 2-form

$$R_{mn}{}^p{}_q(\omega)J^x{}_p{}^q = -2m\kappa J^x{}_{mn}, \quad (\text{B.7})$$

plays the same role as the relation between the Kähler 2-form and the Ricci 2-form in Kähler-Hodge manifolds. It establishes a bridge between symmetric spaces and QK spaces: here, A will play the role of the $H = \text{SU}(2)$ connection ϑ and the hyperKähler structure J^x will play the role of the curvature $R(\vartheta)$ thanks to the above property.

$$A^x \rightarrow \vartheta^x, \quad J^x \rightarrow \frac{1}{\kappa} R^x(\vartheta). \quad (\text{B.8})$$

The fermionic fields of $\mathcal{N} = 2, d = 4$ supergravity are sections of the $\text{SU}(2)$ bundle and, under an transformation with infinitesimal parameter λ^x they transform as⁴³

$$\delta_\lambda \psi^I = \frac{i}{2} \lambda^x \sigma^I{}_J \psi^J, \quad (\text{B.9})$$

and the $\text{SU}(2)$ -covariant derivative acting on them is given by

$$\mathcal{D}\psi^I \equiv d\psi^I - \frac{i}{2} A^x \sigma^I{}_J \psi^J. \quad (\text{B.10})$$

Compare this definition with that of the H-covariant derivative Eq. (1.30).

Now let us assume the existence of an isometry group of the QK manifold preserving the hyperKähler structure J^x . This means that the transformations generated by the corresponding Killing vectors k_A (known as triholomorphic Killing vectors) leave invariant J^x up to an $\text{SU}(2)$ transformation

$$\mathcal{L}_{k_A} J^x = -\delta_{\lambda_A} J^x, \quad \text{or} \quad \mathbb{L}_{k_A} J^x \equiv \mathcal{L}_{k_A} J^x - \varepsilon^{xyz} \lambda_A^y J^z = 0, \quad (\text{B.11})$$

for some $\text{SU}(2)$ infinitesimal parameters λ_A^x which play the role of the H-compensators defined in Eq. (1.27)

$$\lambda_A^x \rightarrow W_A^x. \quad (\text{B.12})$$

In order to determine λ_A^x we observe that the H-compensator has to be *universal*: all the objects that define the QK geometry must be invariant under the action of the isometry and the same compensating $\text{SU}(2)$ transformation. In particular, for the $\text{SU}(2)$ connection

$$\mathbb{L}_{k_A} A^x{}_m = \mathcal{L}_{k_A} A^x{}_m + \mathcal{D}_m \lambda_A^x = k_A{}^n F^x{}_{nm} + \mathcal{D}_m (k_A{}^n A^x{}_n + \lambda_A^x) = 0. \quad (\text{B.13})$$

This equation implies that $k_A{}^n F^x{}_{nm}$ is the $\text{SU}(2)$ -covariant derivative of an object that we can identify with the (tri holomorphic) momentum map:

⁴³Here we ignore the $\text{U}(1)$ component of the R-symmetry group, which we have discussed in the previous Appendix.

$$k_A{}^n F_{nm}^x = \varkappa D_m P_A^x, \quad (\text{B.14})$$

$$\varkappa P_A^x = k_A{}^n A_n^x + \lambda_A^x. \quad (\text{B.15})$$

The last equation should be compared with Eq. (1.27) while the first should be compared with Eq. (1.34). Using the relation between the curvature F^x and the hyperKähler structure J^x Eq. (B.5) one can multiply both sides of the first equation by J^x and obtain

$$k_A{}^m = -\frac{1}{3\varkappa} J^{xmn} D_n P_A^x. \quad (\text{B.16})$$

In this equation the triholomorphic momentum map plays the role of triholomorphic Killing prepotential.

The construction of gauge-covariant derivatives using the momentum map follows the same pattern as in symmetric spaces (see, for instance, Ref. [18]). The momentum map also appears in all the fermion shift terms of the supersymmetry transformation rules of the fermions of the $\mathcal{N} = 2, d = 4$ theories except in those of the hyperinos. Again, the details can be found in Ref. [18].

B.1 2-form potentials from the Quaternionic–Kähler manifolds in $\mathcal{N} = 2, d = 4$ supergravity

Since the hyperscalars do not couple to the vector fields, their dualization is specially simple: the NGZ currents are equal to the Noether current of the σ -model. The supersymmetry transformation rules for the dual 2-forms are given by [45]

$$\delta_\epsilon B_{A\mu\nu} = -4P_A{}^J \bar{\epsilon}^I \gamma_{[\mu} \psi_{J]\nu]} + \frac{8i}{3} U_{\alpha J}{}^u \mathfrak{D}_u P_A{}^J \bar{\epsilon}^I \gamma_{\mu\nu} \zeta^\alpha + \text{c.c.}, \quad (\text{B.17})$$

where u, v label the real coordinates of the QK manifold (the hyperscalars q^u , $U_{\alpha J}{}^u$ are the inverse Vielbein of the QK manifold (the tangent space index being splint into a $SU(2)$ index J and an $Sp(2m)$ index α and where

$$P_A{}^J{}_I \equiv \frac{i}{2} P_A^x (\sigma^x)^I{}_J. \quad (\text{B.18})$$

Again, these supersymmetry transformation rules fit into the general pattern proposed in Section 3, once one takes into account the relation between the derivative of the triholomorphic momentum map and the triholomorphic Killing vectors Eq. (B.14).

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